Mabuchi Geometry of the Space of Kähler/Sasaki Potentials

Thomas Franzinetti (IMJ-PRG)

11 February 2021

Riemannian geometry: metrics on manifolds.

- Looking for special metrics (e.g. Einstein metrics, constant scalar curvature): hard problem in general.
- In real dimension 2, uniformization theorem: *Every surface is a quotient of either the sphere, the euclidean space or the hyperbolic space.*

- Particular cases: on Kähler manifolds, Sasakian manifolds etc.
  - Rich structure.
  - Extensively studied:

Kähler T. Aubin, S.-T. Yau, S. K. Donaldson, X. Chen, S. Sun, G. Tian, J. Cheng etc.

Sasaki D. Martelli, J. Sparks, S.-T. Yau, C. P. Boyer, K. Galicki, T. C. Collins, G. Székelyhidi, P. Guan, X. Zhang, W. He, J. Li etc.

The Kähler Case.

What is a Kähler manifold? (X^n, J, ω)

- Complex manifold ($\dim_{\mathbb{C}} X^n = n$): $J : TX \to TX$ such that

$$J^2 = -Id_{TX}.$$

- Real closed positive $(1,1)$-form: $\omega \underset{loc}{=} i \sum_{\alpha,\beta} h_{\alpha\overline{\beta}} dz_\alpha \wedge d\overline{z}_\beta$.

$$\left(h_{\alpha\overline{\beta}}\right) \text{ positive hermitian matrix.}$$

- $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric.
- $h = g + i\omega$ is a Hermitian metric.

Riemannian, Symplectic, Hermitian + Compatibility + Integrability.

Examples

- Complex Flat space ($n \geq 1$): $(\mathbb{C}^n)$. With $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$
- Complex Projective spaces ($n \geq 1$): $(\mathbb{CP}^n, \omega_{FS})$. On the chart $\{[1 : z_1 : \cdots : z_n]\} \approx \mathbb{C}^n$:

$$h_{\alpha\overline{\beta}} = \frac{\partial^2}{\partial z_\alpha \partial \overline{z_\beta}} \left(\log(1 + |z_1|^2 + \cdots + |z_n|^2\right)$$

- Any complex projective manifold, e.g. zero set of homogeneous polynomials: $\{P = 0\} \subset \mathbb{CP}^n$. The Kähler form is the restriction of ω_{FS}.

Compact Kähler Manifolds.

Instead of searching for metrics, we can search for functions in a certain space.

- A Kähler form defines a $(1,1)$-cohomology class.
- ($\partial\overline{\partial}$-lemma) If two $(1,1)$-forms ω_1 and ω_2 lie in the same cohomology class, then there exists a smooth function ϕ such that:

$$\omega_1 - \omega_2 = i\partial\overline{\partial}\phi.$$

From now on, (X^n, ω) is a compact Kähler manifold.

$$\mathcal{H}_\omega = \left\{\phi \in \mathcal{C}^\infty(X) \mid \omega_\phi = \omega + i\partial\overline{\partial}\phi > 0\right\}.$$

Locally, the condition is: $h_{\alpha\overline{\beta}} + \frac{\partial^2\phi}{\partial z_\alpha \partial \overline{z}_\beta} > 0$ as hermitian matrix.

- $\mathcal{H}_\omega \subset \mathcal{C}^\infty(X)$ open, Fréchet submanifold.
- $T_\phi\mathcal{H}_\omega \approx \mathcal{C}^\infty(X)$.
- Mabuchi's metric (1986): given $\phi \in \mathcal{H}_\omega$, $\psi_1, \psi_2 \in T_\phi\mathcal{H}_\omega \approx \mathcal{C}^\infty(X)$,

$$\langle \psi_1, \psi_2 \rangle_\phi := \int_X \psi_1\psi_2 \, \omega_\phi^n.$$

$$\text{Mabuchi metric } \langle \psi_1, \psi_2 \rangle_\phi := \int_X \psi_1 \psi_2 \, \omega_\phi^n.$$

- Length of path: $[0,1] \ni t \mapsto \phi_t(\cdot)$,

$$L(\phi_t) := \int_0^1 \sqrt{\left\langle \dot{\phi}_t, \dot{\phi}_t \right\rangle_{\phi_t}} dt.$$

- Distance (X. Chen - 2000):

$$d(\phi_0, \phi_1) := \inf \left\{ L(\phi_t) \mid t \mapsto \phi_t(\cdot) \text{ is a path from } \phi_0 \text{ to } \phi_1 \right\}$$

HARD:

$$d(\phi_0, \phi_1) = 0 \quad \overset{???}{\Rightarrow} \quad \phi_0 = \phi_1.$$

$$\text{Mabuchi metric } \langle \psi_1, \psi_2 \rangle_\phi := \int_X \psi_1 \psi_2 \, \omega_\phi^n.$$

- Natural connexion of non-positive sectional curvature.
- Geodesic equation:

$$\ddot{\phi}_t - \frac{1}{2} g_{\phi_t}(\nabla \dot{\phi}_t, \nabla \dot{\phi}_t) = 0$$

$$\boxed{g_{\phi_t} \longleftrightarrow \omega_{\phi_t} \text{ AND } \nabla = \nabla^{g_{\phi_t}}}$$

- Example on $\mathbb{CP}^1$ (in a chart):

$$t \mapsto \phi_t(z) := \log(1 + e^{2t}|z|^2) - 2\log(1 + |z|^2).$$

- No existence in $\mathcal{H}_\omega$ in general (L. Lempert - L. Vivas).

What is a geodesic in $\mathcal{H}_\omega$?

$$\ddot{\phi}_t - \frac{1}{2} g_{\phi_t}(\nabla \dot{\phi}_t, \nabla \dot{\phi}_t) = 0 \qquad \text{(WGE)}$$

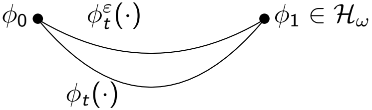
- Monge-Ampère reformulation (S. K. Donaldson, S. Semmes):

$$\text{(WGE)} \leftrightarrow \text{(MA)}$$

- (MA) makes sense for bounded functions.
- (MA) has a unique solution: *weak solution*.
- X. Chen (2000):

$$\forall t \in [0,1]\,, \quad d(\phi_0, \phi_1)^2 = \int_X |\dot{\phi}_t|^2 \omega^n_{\phi_t}. \quad \Rightarrow \text{metric geodesic}$$

We want to approximate weak geodesics by smooth paths.

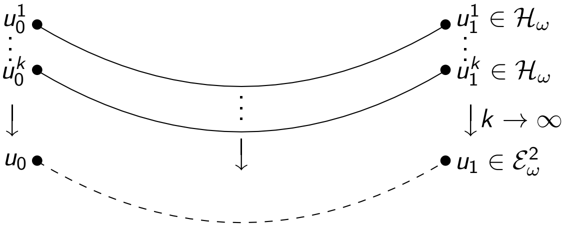


- ε-geodesics equation: ε-perturbation of (WGE).
- In the same spirit:
$$\text{(WGE)}_\varepsilon \leftrightarrow \text{(MA)}_\varepsilon.$$
- Unique smooth solution.
- Decrease to weak geodesics: for any t,

$$\phi_t^\varepsilon(\cdot) \xrightarrow[\varepsilon \to 0]{\mathcal{C}^{1,\bar{1}}} \phi_t(\cdot) \qquad \Rightarrow \phi_t \in \mathcal{C}^{1,\bar{1}}.$$

The space $\mathcal{H}_\omega$ is not complete: WANT to understand the geometry of its metric completion: $\mathcal{E}^2_\omega$.

What is a geodesic in $\mathcal{E}^2_\omega$? What is the distance ?



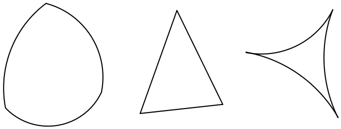
- Extension of Mabuchi's metric: Darvas distance $\tilde{d}$ on $\mathcal{E}^2_\omega$.
- $\mathcal{E}^2_\omega$ is a geodesically complete metric space (T. Darvas 2014).

Question: What is the curvature of $(\mathcal{E}^2_\omega, \tilde{d})$?

- Sectional curvature computed for smooth potentials.
- Alexandrov notion of curvature: CAT(0).

What does that mean ?

- Notion of curved geodesic *metric* spaces:

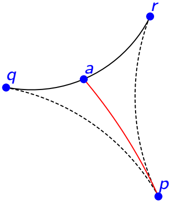


- Examples: simply connected complete Riemannian manifolds of non-positive sectional curvature; hyperbolic space, euclidean space, tree etc.

More precisely...

CAT(0) inequality: $\lambda = \frac{d(q,a)}{d(q,r)} < 1$,

$$d(p,a)^2 \leq \lambda d(p,r)^2 + (1-\lambda)d(p,q)^2 - \lambda(1-\lambda)d(q,r)^2. \quad \text{(CAT(0))}$$



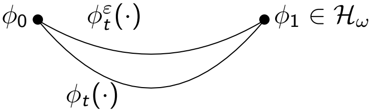
If $\lambda = \frac{1}{2}$, then: $d(p,a)^2 \leq \frac{1}{2}d(p,r)^2 + \frac{1}{2}d(p,q)^2 - \frac{1}{4}d(q,r)^2$ (Apollonius).

Theorem (T. Darvas 2014)

The metric completion, $\mathcal{E}^2_\omega$, of $(\mathcal{H}_\omega, d)$ is non-positively curved in the sense of Alexandrov (it is a CAT(0) space).

First, we prove the inequality on $\mathcal{H}_\omega$.

Tools for the proof:

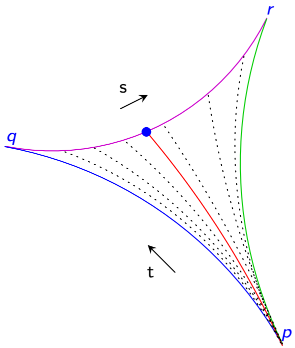


- Integral expression for the distance. Let $\phi_t^\varepsilon(\cdot)$ be the ε-geodesic joining $\phi_0(\cdot), \phi_1(\cdot) \in \mathcal{H}_\omega$,

$$d(\phi_0, \phi_1)^2 = \int_X |\dot{\phi}_t|^2 \omega_{\phi_t}^n \overset{\text{Chen}}{=} \lim_{\varepsilon \to 0} \underbrace{\int_0^1 \int_X \left(\dot{\phi}_t^\varepsilon\right)^2 \omega_{\phi_t^\varepsilon}^n \, dt}_{\text{energy}} =: \lim_{\varepsilon \to 0} E^\varepsilon.$$

- The sectional curvature in $\mathcal{H}_\omega$ (smooth) is non-positive.

Steps of the proof $\simeq$ Convexity of the "distance" ($\dim < \infty$).



Let $p, q, r \in \mathcal{H}_\omega$,

- ε-geodesic from q to r.
- Two parameters map: $\phi^\varepsilon(\cdot, t, s) \in \mathcal{H}_\omega$.
- Energy of the ε-geodesic: $E^\varepsilon(s) = \int_0^1 \int_M \left(\frac{\partial \phi^\varepsilon}{\partial t}\right)^2 \omega^n_{\phi^\varepsilon} dt$
- Sectional curvature ≤ 0 gives "ε-Convexity" of $E^\varepsilon(s)$:

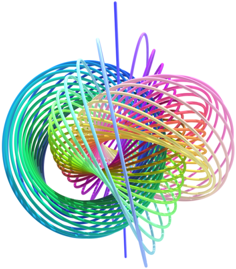
$$E^\varepsilon(s) \leq (1-s)E^\varepsilon(0) + sE^\varepsilon(1) - s(1-s)\left(E^\varepsilon + o(\varepsilon)\right)$$

$$\tilde{d}(p, \bullet) \leq (1-s)d(p,q) + sd(p,r) - s(1-s)d(q,r)$$

Finally: Approximation of $\mathcal{E}^2_\omega$ with decreasing sequences in $\mathcal{H}_\omega$.

The Sasakian Case.

Example: the Hopf fibration.



$$\begin{array}{ccccc} & & (\mathbb{C}^2, \text{flat}) & & \textit{Kähler} \\ & & \downarrow & & \\ \mathbb{S}^1 & \longrightarrow & (\mathbb{S}^3, g = \text{round}) & & \text{Sasaki} \\ & & \downarrow \pi & & \\ & & (\mathbb{CP}^1, \omega_{FS}, J) & & \textit{Kähler} \end{array}$$

- Isometric $\mathbb{S}^1$-action.
- ξ tangent to the fibre and η dual 1-form (i.e. $\eta(\xi) = 1$ and $\iota_\xi d\eta = 0$).
- Kähler Horizontal structure: $\frac{1}{2} d\eta = \pi^\star \omega_{FS}$ and Φ from J.

The tensors (ξ, η, Φ, g) "talk" to each other.

$$(M^{2n+1}, \xi, \eta, \Phi, g)$$

Riemannian, Contact, Almost Complex + Compatibility + Integrability.

Nice characterization: If the metric cone $(C(M), dr^2 + r^2 g)$ is Kähler, then M is Sasakian.

$$\underbrace{C(M)}_{\text{Kähler cone.}} \longrightarrow M \longrightarrow \underbrace{M/\mathcal{F}_\xi}_{\text{"Kähler"}}$$

Trapped between two Kähler: "Odd dimensional Kähler analogue".

The Space of Sasakian potentials (P. Guan - X. Zhang 2009):

- Basic functions $\mathcal{C}^{\infty}_B(M)$: ξ-invariance.
- Transverse Kähler structure: transverse differential operators AND transverse $\partial_B\overline{\partial}_B$-lemma.

$$\mathcal{H}(M,\xi,\eta) = \left\{\phi \in \mathcal{C}^{\infty}_B(M),\ d\eta + i\partial_B\overline{\partial}_B\phi > 0\right\}.$$

Write $\eta_\phi := \eta + \frac{i}{2}\left(\partial_B - \overline{\partial}_B\right)\phi$, so that $d\eta_\phi = d\eta + i\partial_B\overline{\partial}_B\phi$.

$$\phi \in \mathcal{H}(M,\xi,\eta) \longrightarrow (\xi, \eta_\phi, \Phi_\phi, g_\phi) \text{ Sasakian structure.}$$

Here,

- $\Phi_\phi := \Phi - \xi \otimes (d^c\phi \circ \Phi)$
- $g_\phi := \eta_\phi \otimes \eta_\phi + \frac{1}{2}d\eta(1_{TM} \otimes \Phi_\phi)$

$$\mathcal{H}(M,\xi,\eta) = \left\{\phi \in \mathcal{C}^\infty_B(M),\ d\eta + i\partial_B\overline{\partial}_B\phi > 0\right\}.$$

- Mabuchi like metric, non-positive sectional curvature (GZ09):

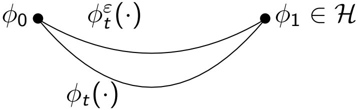
$$\langle\psi_1,\psi_2\rangle_\phi = \int_M (\psi_1\psi_2)\eta_\phi \wedge d\eta_\phi^n.$$

- Geodesic equation:

$$\ddot{\phi}_t - \frac{1}{4}g_{\phi_t}(\nabla\dot{\phi}_t,\nabla\dot{\phi}_t) = 0.$$

- Monge-Ampère reformulation.
- ε-geodesics: perturbated equation.

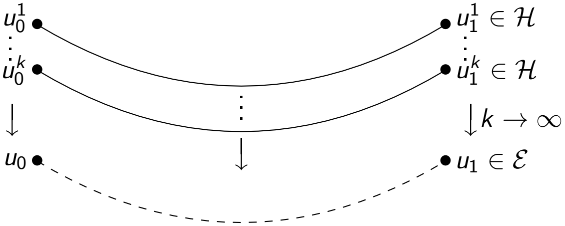
Distance on $\mathcal{H}(M, \xi, \eta)$ (GZ09):



$$d(\phi_0, \phi_1)^2 = \int_M |\dot{\phi}_t|^2 \eta_{\phi_t} \wedge d\eta^n_{\phi_t} = \lim_{\varepsilon \to 0} \underbrace{\int_0^1 \int_X \left(\dot{\phi}_t^\varepsilon\right)^2 \eta_{\phi_t^\varepsilon} \wedge d\eta^n_{\phi_t^\varepsilon} \, dt}_{\text{energy}} .$$

The metric completion (W. He - J. Li 2018).

- Pluripotential theory $\to$ space $\mathcal{E}^2$.
- The metric completion of $\mathcal{H}$ is $\mathcal{E}^2$.
- Extension of the distance.
- Metric geodesic.



Once again, we want to understand the geometry of this space: can we extend T. Darvas theorem to the Sasakian setting ?

Recall that the sectional curvature of $\mathcal{H}(M, \xi, \eta)$ is non-positive (GZ09).

Theorem (F. 2019)

The metric completion of $\mathcal{H}(M, \xi, \eta)$ is a CAT(0)-space.

⚠ It is worth to mention that not all the results established in the Kähler case can be straightforwardly adapted to the Sasakian world.

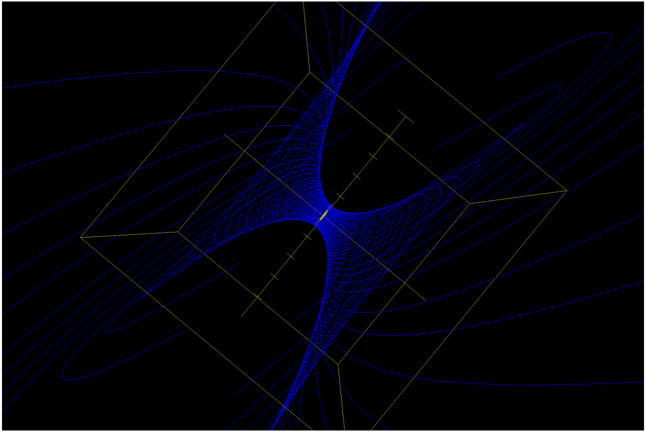
It is Darvas theorem for more general Kähler structures. Example on the *modified* Hopf action:

$$t \mapsto \left(e^{it} z_1, e^{i\alpha t} z_2\right), \quad \alpha \in \mathbb{Z}.$$

The quotient is then:

$$\begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & (\mathbb{S}^3, g = \text{round}) \\ & & \big\downarrow \pi \\ & & \text{Football Orbifold} \end{array}$$

$$t \mapsto \left(e^{it} z_1, e^{i\alpha t} z_2\right), \quad \alpha \in \mathbb{R} \backslash \mathbb{Q}.$$



Thank you for your attention.