Equivariant Cohomology of GKM Manifolds

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Outline

- Hamiltonian actions on symplectic manifolds
- GKM manifolds
- GKM graphs/one-skeleta and why they're interesting
- Equivariant cohomology
- Equivariant cohomology of GKM manifolds/graphs/one-skeleta
- Running example: $\mathbb{P}^3$
- (Tolman's manifold)

Symplectic Manifolds

Definition

A *symplectic manifold* is a pair (M, ω) where

- M is a smooth manifold with $\dim_{\mathbb{R}}(M) = 2d$
- The *symplectic form*, ω, is a closed, non-degenerate 2-form.

Theorem (Darboux)

For every point x in (M, ω), there exists a system of local coordinates $(p_1, \ldots, p_d, q_1, \ldots, q_d)$ centred at x such that $\omega = \omega_0 = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$.

Theorem ('Equivariant Darboux')

Let G be a compact Lie group acting on (M, ω) where ω is G-invariant. If $p \in M^G$ is a fixed point of the action then, with respect to a linear action of G on $\mathbb{R}^{2d}$, there exists a system of G-equivariant local coordinates centred at p in which $\omega = \omega_0$.

Hamiltonian Actions

Definition

Let G be a Lie group and (M, ω) a symplectic manifold. We say the action $\tau : G \times M \to M$ is *Hamiltonian* if there exists a map $\mu : M \to \mathfrak{g}^*$ which satisfies:

1. For each $\xi \in \mathfrak{g}$, writing
    - $\mu^\xi : M \to \mathbb{R}$, $\mu^\xi(x) = \langle \mu(x), \xi \rangle$, for the component of μ along ξ,
    - ξ_M for the vector field on M generated by $\{\exp t\xi \mid t \in \mathbb{R}\} \subseteq G$,

   then μ^ξ is a Hamiltonian function for the vector field ξ_M;

   $$\iota(\xi_M)\omega = -d\mu^\xi.$$

2. The map μ is G-equivariant with respect to the action τ on M and the coadjoint action Ad^* on $\mathfrak{g}^*$.

We say μ is a *moment map* and (M, ω, G, μ) is a *Hamiltonian G-space*.

Set-up

Unless otherwise stated

- G will be a commutative, compact, connected, n-dimensional Lie group (i.e. $(S^1)^n$) with corresponding Lie algebra $\mathfrak{g}$.
- M is a compact $2d$-dimensional manifold with a faithful G-action $\tau : G \times M \to M$.

Given $p \in M$, for each element of the isotropy group $g \in G_p$ we restrict the action, $\tau_g : M \to M$, and take the derivative at p to define

$$\rho : G_p \to GL(T_pM),\ \rho(g) = (\mathrm{d}\tau_g)_p$$

called the *isotropy representation*.
Note that if $p \in M^G$ then ρ is an action of the whole group G on T_pM.

We are interested in the weights of this representation.

GKM manifolds

Definition

The manifold M is a *GKM manifold* if it satisfies the following:

1. The fixed point set M^G is finite.
2. There is a G-invariant almost-complex structure on M.
3. For each fixed point $p \in M^G$, the weights of the isotropy representation of G on T_pM,

   $$\alpha_{j,p} \in \mathfrak{g}^*, \ j = 1, \ldots d$$

   are pairwise linearly independent.

What do they look like?

Let $p \in M^G$ be a fixed point. For each weight of the isotropy representation at p,

$$\alpha_{j,p} \in \mathfrak{g}^*, \ j = 1, \dots d$$

let $\mathfrak{h}_j$ denote the annihilator of $\alpha_{j,p}$ in $\mathfrak{g}$.

Let H_j be the $(n-1)$-dimensional subtorus of G which has $\mathfrak{h}_j$ as its Lie algebra and X_j the connected component of M^{H_j} containing p.

Proposition

For each j, X_j is diffeomorphic to S^2 and the the action of G on X_j is equivalent to the standard action of the circle G/H_j on S^2 by rotation. In particular X_j has exactly two G-fixed points.

What do they look like?

- The fixed point p is the intersection point of d embedded G-invariant 2-spheres.
- Since the standard action of S^1 on S^2 has two fixed points, each sphere connects p to another fixed point $q_i \in M^G$, $i = 1, \ldots, d$.
- Similarly q_i is the intersection point of d embedded G-invariant 2-spheres, one of which will be X_i, the sphere connecting p and q_i.

We use a graph to express this.

The fixed points and spheres are described by the vertices and edges respectively.

It follows that each vertex has degree d - the graph is d-valent.

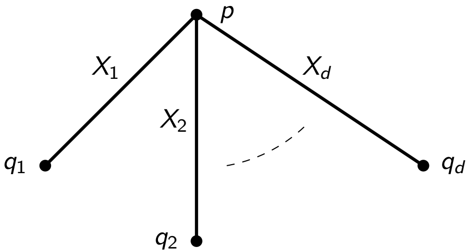


Figure: GKM graph of M at p

GKM one-skeleta

Such a graph is called the *GKM graph* of M and denoted by Γ.
We write E_Γ for the set of directed edges.

Definition

The *axial function* of a GKM graph Γ is a map

$$\alpha : E_\Gamma \to \mathfrak{g}^*, \ e \mapsto \alpha_e$$

where α_e is the weight of the isotropy representation of G on $T_{i(e)}X_e$.

We will often use the axial function as a labelling of the directed edges to keep note of the G action.

Definition

We call the pair (Γ, α) the *GKM one-skeleton* of the GKM manifold M.

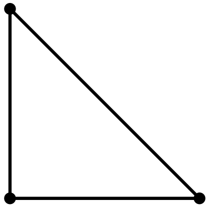


Figure: GKM graph of $\mathbb{P}^2$

Intuition

Let M be a GKM manifold.
By definition we have a G-invariant almost-complex structure on M, say J.
Let g be a compatible G-invariant metric. Using these we define

$$\omega(u, v) = g(Ju, v)$$

a G-invariant almost-symplectic structure (a non-degenerate 2-form).

Additionally if we suppose that ω is closed, then (M, ω) is a Hamiltonian G-space with moment map μ.

Theorem

$\mu(M)$ is a convex polytope. The vertices are the images of the fixed points $p \in M^G$ and the primitive vectors along the edges emanating from $\mu(p)$ are the vectors $\alpha_{j,p}$.

The *moment graph* of M is the one-skeleton of $\mu(M)$ and coincides with the GKM graph of M.

Example

Consider $(\mathbb{P}^3, 2\omega_{FS})$ with the standard $\mathbb{T}^3$-action

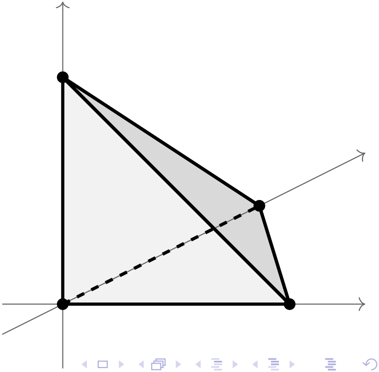
$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2 : e^{i\theta_3} z_3].$$

The moment map is given by

$$\mu : \mathbb{P}^3 \to \mathbb{R}^3,\ \mu[z_0 : z_1 : z_2 : z_3] = \left(\frac{|z_1|^2}{\sum_{j=0}^3 |z_j|^2}, \frac{|z_2|^2}{\sum_{j=0}^3 |z_j|^2}, \frac{|z_3|^2}{\sum_{j=0}^3 |z_j|^2} \right)$$

The four fixed points are mapped to the vertices of the moment polytope:

$$[1 : 0 : 0 : 0] \mapsto (0, 0, 0)$$
$$[0 : 0 : 1 : 0] \mapsto (0, 1, 0)$$
$$[0 : 1 : 0 : 0] \mapsto (1, 0, 0)$$
$$[0 : 0 : 0 : 1] \mapsto (0, 0, 1)$$



Example

The closures of the six one-dimensional orbits along with their corresponding isotropy groups:

$[* : * : 0 : 0]$	$\{(1, t, t) \mid t \in S^1\}$
$[* : 0 : * : 0]$	$\{(t, 1, t) \mid t \in S^1\}$
$[* : 0 : 0 : *]$	$\{(t, t, 1) \mid t \in S^1\}$
$[0 : * : * : 0]$	$\{1\} \times \{1\} \times S^1$
$[0 : * : 0 : *]$	$\{1\} \times S^1 \times \{1\}$
$[0 : 0 : * : *]$	$S^1 \times \{1\} \times \{1\}$

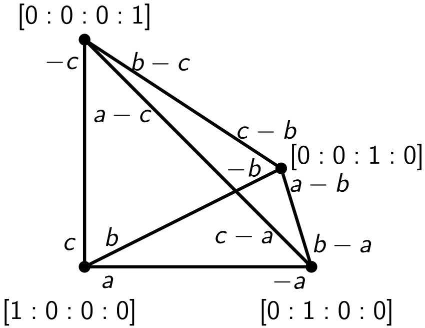


Figure: GKM one-skeleton of $\mathbb{P}^3$

Properties of one-skeleta: we can define connections, holonomy, geodesic subgraphs, ...

Betti numbers

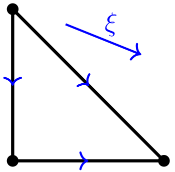
Definition

$\xi \in \mathfrak{g}$ is a *polarising vector* if $\langle \alpha_e, \xi \rangle \neq 0$ for all directed edges $e \in E_\Gamma$.

Directing of each edge e of Γ so that $\langle \alpha_e, \xi \rangle > 0$ gives the *ξ-orientation* o_ξ.

Definition

Let $\xi \in \mathfrak{g}$ be a polarising vector.
The *index σ_p of a vertex p* is the number of edges e of the directed graph (Γ, o_ξ) which terminate at p.



Definition

The *(combinatorial) $2i$-th Betti number of* (Γ, o_ξ) is the number of vertices of Γ with exactly i negative weights;

$$b_{2i}(\Gamma) = \#\{p \in V_\Gamma \mid \sigma_p = i\}.$$

Betti numbers

Proposition

The Betti numbers $b_{2i}(\Gamma)$ *are combinatorial invariants of the GKM one-skeleton* (Γ, α).

Proposition

If the G*-action on* (M, ω) *is Hamiltonian then* $b_{2i}(\Gamma) = b_{2i}(M)$.

Idea of proof.

Use the equivariant Darboux theorem to show that the projection of the moment map

$$\langle \mu, \xi \rangle : M \to \mathbb{R}$$

is perfect Morse function. □

Can we also read off the structure of the cohomology ring?
What about equivariant cohomology?

Equivariant Cohomology

Let G be a Lie group and recall the coadjoint representation of G on $\mathfrak{g}^*$; for $\alpha \in \mathfrak{g}^*$, $\xi \in \mathfrak{g}$

$$\langle \mathrm{Ad}_g^* \alpha, \xi \rangle = \langle \alpha, \mathrm{Ad}_{g^{-1}}(\xi) \rangle.$$

The symmetric algebra on $\mathfrak{g}^*$, $\mathbb{S}(\mathfrak{g}^*)$, may be thought of as the algebra of polynomials on $\mathfrak{g}$ and there is a natural extension of Ad_g^* to $\mathbb{S}(\mathfrak{g}^*)$.

We denote by $\mathbb{S}(\mathfrak{g}^*)^G$ the subspace of G-invariant polynomials, that is polynomials constant along adjoint orbits in the Lie algebra.

Remark

If G is compact and connected then $\mathbb{S}(\mathfrak{g})^*$ is also a polynomial ring.

Let G be compact and act on a manifold M, with $(\Omega(M), d)$ the usual de Rham complex of differential forms on M.
Consider $\mathbb{S}(\mathfrak{g}^*) \otimes \Omega(M)$ with the tensor product representation; G acts on $\mathbb{S}(\mathfrak{g}^*)$ by the coadjoint representation and on $\Omega(M)$ by the pullback of forms; $g \cdot \eta = (g^{-1})^* \eta$.

Equivariant Cohomology

Definition

The space of *equivariant differential forms on M* is the subspace of G-invariant objects

$$\Omega_G(M) = (\mathbb{S}(\mathfrak{g}^*) \otimes \Omega(M))^G.$$

From an equivariant form

$$\omega = \sum f_i \otimes \eta_i \in \Omega_G(M), \text{ with } f_i \in \mathbb{S}(\mathfrak{g}^*), \ \eta_i \in \Omega(M)$$

we build an associated polynomial map $\omega : \mathfrak{g} \to \Omega(M), \ \xi \mapsto \sum f_i(\xi)\eta_i$. This map is G-equivariant, and allows us to think of $\Omega_G(M)$ as the space of G-equivariant polynomial maps $\mathfrak{g} \to \Omega(M)$.

Remark

For GKM spaces $G = (S^1)^n$ so the (co)adjoint action is trivial giving

$$\Omega_G(M) = \mathbb{S}(\mathfrak{g}^*) \otimes \Omega(M)^G.$$

The G-equivariant forms are polynomials $\omega : \mathfrak{g} \to \Omega(M)^G$.

Equivariant Cohomology

Note that $\Omega_G(M)$ is a ring with respect to the wedge product with grading

$$\Omega_G^m(M) = \bigoplus_{2k+l=m} (\mathbb{S}^k(\mathfrak{g}^*) \otimes \Omega^l(M))^G$$

Definition

Let $\{\xi_i\}$ denote a basis of $\mathfrak{g}$ and $\{\mu_i\}$ the dual basis of $\mathfrak{g}^*$.
The *Cartan differential* $d_G : \Omega_G^m(M) \to \Omega_G^{m+1}(M)$ is given by

$$d_G = 1 \otimes d - \sum \mu_i \otimes \iota_{(\xi_i)_M}$$

or from the point of view of polynomial maps: $d_G\omega(\xi) = d\omega(\xi) - \iota_{\xi_M}\omega(\xi)$.

Definition

The *equivariant cohomology* of M is given by $H_G^*(M) = H_{dR}^*(\Omega_G^*(M), d_G)$.

Equivariant Formality

We may consider $\Omega_G(M)$ as a double complex with grading

$$\Omega_G^{p,q}(M) = (\mathbb{S}^p(\mathfrak{g}^*) \otimes \Omega^{q-p}(M))^G$$

where the respective vertical and horizontal operators are the first and second summands of the Cartan differential d_G.

Definition

We say M is *equivariantly formal* with respect to the action of G if the spectral sequence of the Cartan complex collapses at the E_1 term.

Equivariant Formality

Proposition

If $H^{2k+1}(M) = 0$ for all k, then the G-action on M is equivariantly formal.

Recall: a 'nice' projection of the moment map is a perfect Morse function.

Corollary

A Hamiltonian G-action on M is equivariantly formal.

Theorem

If M is equivariantly formal we have an isomorphism of $\mathbb{R}$-algebras

$$H^*(M) \cong \frac{H_G^*(M)}{J \cdot H_G^*(M)}$$

where J denotes the augmentation ideal in $\mathbb{S}(\mathfrak{g}^*)$.

Graph Cohomology

Definition

Let $V_\Gamma = \{p_1, \ldots, p_N\}$ then the *cohomology ring of* (Γ, α) is

$$H^*(\Gamma, \alpha) = \left\{ \left(f(p_1), \ldots, f(p_N)\right) \in \bigoplus \mathbb{S}(\mathfrak{g}^*) \;\middle|\; \begin{array}{c} f(p_i) - f(p_j) \in \langle \alpha_e \rangle \\ \forall e = p_i p_j \in E_\Gamma \end{array} \right\}$$

where $\langle \alpha \rangle$ denotes the principal ideal generated by α.

Theorem (GKM)

Let M be an equivariantly formal GKM manifold with one-skeleton (Γ, α)*, then*

$$H_G^*(M) \cong H^*(\Gamma, \alpha).$$

Computing Generators

Definition

Let $\xi \in \mathcal{P}$ be a polarising vector and $p \in V_\Gamma$ a vertex.
The *flow-out of* p, F_p, is the set of vertices q of (Γ, o_ξ) such that there exists a directed path from p to q compatible with the ξ-orientation o_ξ.

Proposition (Guillemin–Zara)

Let $p \in V_\Gamma$ be a vertex of (Γ, o_ξ) of index k. Then there exists an element $\tau_p \in H^{2k}(\Gamma, \alpha)$ satisfying

1. *τ_p is supported on the flow-out of p, F_p,*
2. *$\tau_p(p) = \prod \alpha_e$, with the product over directed edges terminating at p.*

If $\{\tau_p\}_{p \in V_\Gamma}$ satisfy these conditions then $H^*(\Gamma, \alpha)$ is a free $\mathbb{S}(\mathfrak{g}^*)$-module generated by $\{\tau_p\}_{p \in V_\Gamma}$.

If $\sigma_q > \sigma_p$ for every $q \in F_p \setminus \{p\}$ then τ_p is unique.

Algorithm for Computing Generators

For $\xi \in \mathcal{P}$ a polarising vector, take the partial ordering on the vertices induced by the index; $p < q$ if $\sigma_p < \sigma_q$. Then we compute τ_p inductively:

- First set $\tau_p(p') = 0$ for all $p' < p$.
- Next take the product
$$\tau_p(p) = \prod \alpha_e,$$
over all edges terminating at p, or set $\tau_p(p) = 1$ if $\sigma_p = 0$.
- Finally if q is a vertex such that we have defined $\tau_p(p')$ for every neighbouring vertex $p' < q$, then we take $\tau_p(q) \in \mathbb{S}(\mathfrak{g}^*)$ to be an element of minimal degree which satisfies the following for each neighbouring $p' < q$;
$$\tau_p(q) - \tau_p(p') \in \langle \alpha_e \rangle, \text{ where } e = p'q.$$

We write a generator τ_p as a labelling of the vertices of Γ, or as follows; if $p_1, \dots, p_N$ is vertex ordering compatible with the partial ordering above, then
$$\tau_p = \big(\tau_p(p_1), \dots, \tau_p(p_N)\big) \in H^{2k}(\Gamma, \alpha).$$

Example

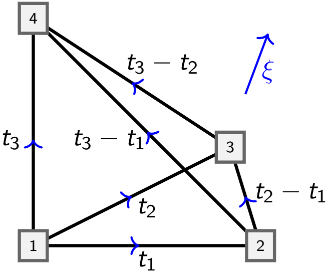
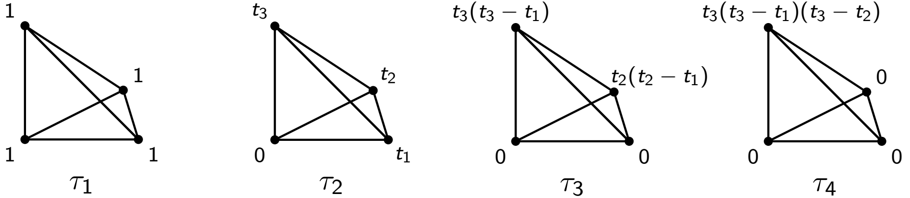


Figure: One-skeleton of $\mathbb{P}^3$

In the other notation, our generators are:

$$\tau_1 = (1, 1, 1, 1) \in H^0_{\mathbb{T}^3}(\mathbb{P}^3)$$
$$\tau_2 = (0, t_1, t_2, t_3) \in H^2_{\mathbb{T}^3}(\mathbb{P}^3)$$
$$\tau_3 = (0, 0, t_2(t_2 - t_1), t_3(t_3 - t_1)) \in H^4_{\mathbb{T}^3}(\mathbb{P}^3)$$
$$\tau_4 = (0, 0, 0, t_3(t_3 - t_1)(t_3 - t_2)) \in H^6_{\mathbb{T}^3}(\mathbb{P}^3)$$

Following the algorithm we obtain the set of generators:



Example

We have component-wise multiplication, so for example

$$\tau_2^2 = (0, t_1^2, t_2^2, t_3^2) = t_1(0, t_1, t_2, t_3) + (0, 0, t_2(t_2 - t_1), t_3(t_3 - t_1)) = t_1\tau_2 + \tau_3.$$

In this way we obtain the multiplication table:

	τ_1	τ_2	τ_3	τ_4
τ_1	τ_1	τ_2	τ_3	τ_4
τ_2	τ_2	$t_1\tau_2 + \tau_3$	$t_2\tau_3 + \tau_4$	$t_3\tau_4$
τ_3	τ_3	$t_2\tau_3 + \tau_4$	$t_2(t_2 - t_1)\tau_3 + (t_3 + t_2 - t_1)\tau_4$	$t_3(t_3 - t_1)\tau_4$
τ_4	τ_4	$t_3\tau_4$	$t_3(t_3 - t_1)\tau_4$	$t_3(t_3 - t_1)(t_3 - t_2)\tau_4$

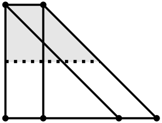
The multiplication table for the usual cohomology ring is as expected; $H^*(\mathbb{P}^3)$ is generated by an element τ_2 of degree 2 such that $\tau_2^4 = 0$:

	τ_1	τ_2	τ_3	τ_4
τ_1	τ_1	τ_2	τ_3	τ_4
τ_2	τ_2	τ_3	τ_4	0
τ_3	τ_3	τ_4	0	0
τ_4	τ_4	0	0	0

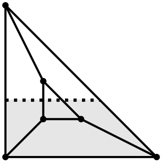
Tolman's manifold

A minimal Hamiltonian $\mathbb{T}^k$-manifold with no compatible $\mathbb{T}^k$-invariant Kähler metric: a six-dimensional Hamiltonian $\mathbb{T}^2$-manifold, $M_{\mathcal{T}}$, with a family of symplectic forms for which there does not exist any compatible $\mathbb{T}^2$-invariant Kähler metric.

$\hat{M}$: take a suitable $\mathbb{T}^2$-action on $\mathbb{C}P^1 \times \mathbb{C}P^2$ and a $\mathbb{T}^3$-invariant symplectic form such that the moment map $\hat{\mu} : \hat{M} \to \mathbb{R}^2$ has image:



$\tilde{M}$: take the projectivisation of the bundle $\mathcal{O} \oplus \mathcal{O}(-3)$ over $\mathbb{P}^2$. It has a natural $\mathbb{T}^3$-action so we choose a suitable $\mathbb{T}^3$-invariant symplectic form and subtorus $\mathbb{T}^2 \subset \mathbb{T}^3$ such that the moment map $\tilde{\mu} : \hat{M} \to \mathbb{R}^3$ has image:



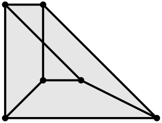
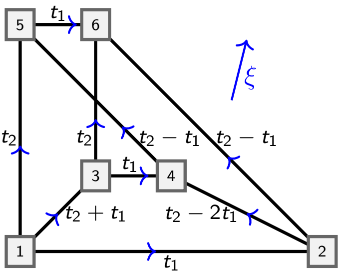


Figure: $\mu(M_{\mathcal{T}})$

Tolman's manifold



	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_1	τ_1	τ_2	τ_3	τ_4	τ_5	τ_6
τ_2	τ_2	τ_5	$\tau_4 - \tau_5$	τ_6	0	0
τ_3	τ_3	$\tau_4 - \tau_5$	$-3\tau_4 + 2\tau_5$	$-2\tau_6$	τ_6	0
τ_4	τ_4	τ_6	$-2\tau_6$	0	0	0
τ_5	τ_5	0	τ_6	0	0	0
τ_6	τ_6	0	0	0	0	0

$$H^*(M_{\mathcal{T}}) \cong \mathbb{Z}[u,v]/(u^2 + 3uv + v^2,\ u^3)$$

where $u = \tau_2,\ v = \tau_3$

$$\tau_1 = (1,1,1,1,1,1) \in H^0_{\mathbb{T}^2}(M_{\mathcal{T}})$$
$$\tau_2 = (0, t_1, 0, t_1, t_2, t_2) \in H^2_{\mathbb{T}^2}(M_{\mathcal{T}})$$
$$\tau_3 = (0, 0, t_2 + t_1, t_2 - 2t_1, -t_2, -(t_2 - t_1)) \in H^2_{\mathbb{T}^2}(M_{\mathcal{T}})$$
$$\tau_4 = (0,0,0, t_1(t_2 - 2t_1), -t_2 t_1, 0) \in H^4_{\mathbb{T}^2}(M_{\mathcal{T}})$$
$$\tau_5 = (0,0,0,0, t_2(t_2 - t_1), t_2(t_2 - t_1)) \in H^4_{\mathbb{T}^2}(M_{\mathcal{T}})$$
$$\tau_6 = (0,0,0,0,0, t_2 t_1(t_2 - t_1)) \in H^6_{\mathbb{T}^2}(M_{\mathcal{T}})$$

Tolman's manifold

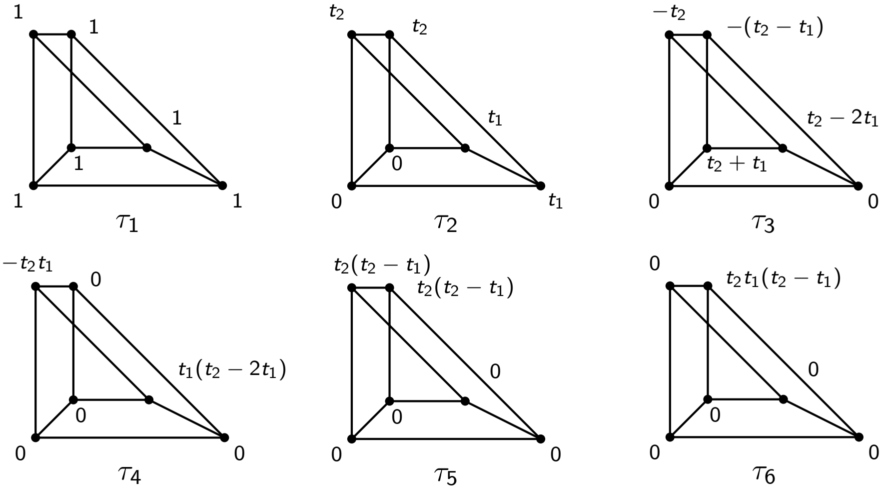


Figure: Generators for the equivariant cohomology ring of Tolman's manifold