The Conway knot is not slice

05 October 2020 16:16

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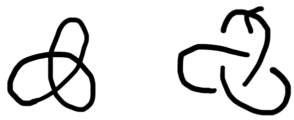
§1 Mutation and sliceness

— Knots

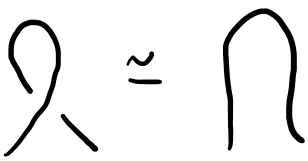
Def. A knot is an embedding $S^1 \hookrightarrow S^3$ or $\mathbb{R}^3$. considered up to ambient isotopy.

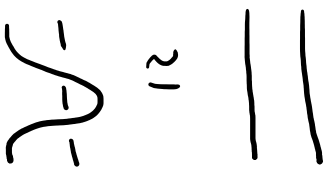
$K \simeq K'$

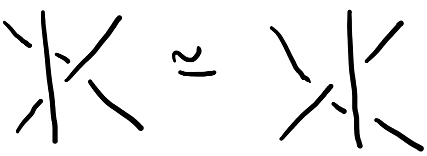


+ve   -ve.

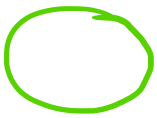
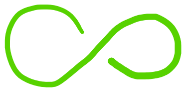
· Reidemeister moves

I 

II 

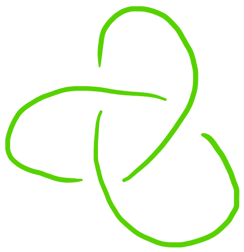
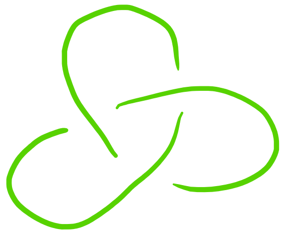
III 

e.g. Unknot, U

 $\underset{I}{\cong}$ 

e.g. Trefoil knots

knot invariant

 $\not\cong$ 

RH LH

e.g. Figure 8 knot

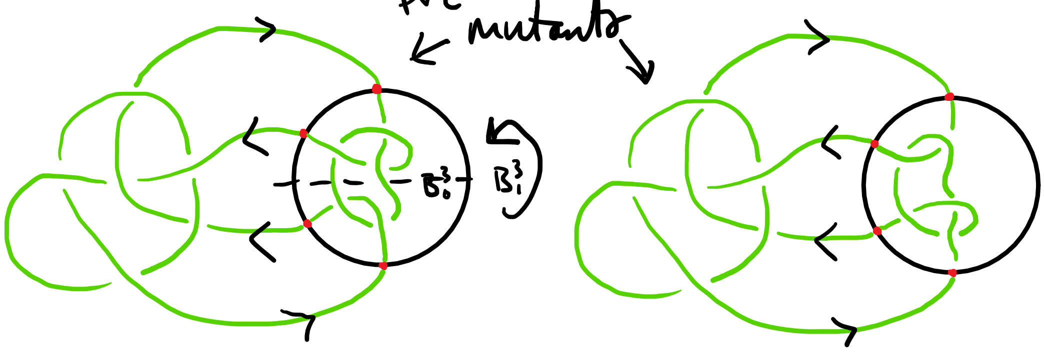
$K \cong \overline{K}$

"amphichiral"

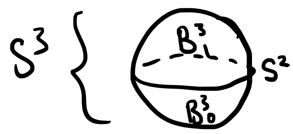
- Mutation

e.g. Conway knot Kinoshita-Terasaka knot

+ve mutants



Def. A Conway sphere for a knot K is an embedded $S^2 \hookrightarrow S^3$ which intersects K transversely in exactly 4 points.



$$S^3 = B^3_0 \underset{S^2}{\cup} B^3_1$$
$$K = K_0 \cup K_1 \qquad \text{"tangles"}$$

Def. K^* is a mutant of K if it can be obtained from K_0 & K_1 by regluing B^3_0 & B^3_1 via an involution of the Conway sphere.

Def. K^* is a positive mutant of K if it is a mutant of K which also inherits a well-defined orientation from K_0 & K_1.

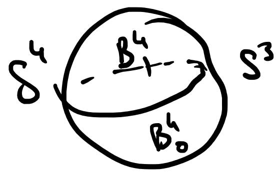
- Positive mutation preserves many 3-dimensional knot invariants.

e.g. Alex. / Jones / HOMFLY polynomials.
e.g. $S^3 \setminus \nu(K)$ hyperbolic volume

— Sliceness

- S^3  u

- S^4 Any knot bounds a disc.

- B^4 

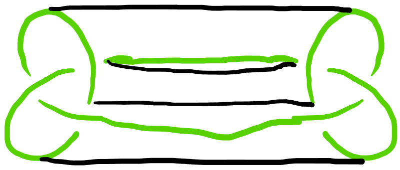
Def. Let K be a knot and suppose $\exists D^2 \subset B^4$ s.t. $K = \partial D^2$.

- K is called topologically slice if $D^2 \subset B^4$ is locally flat

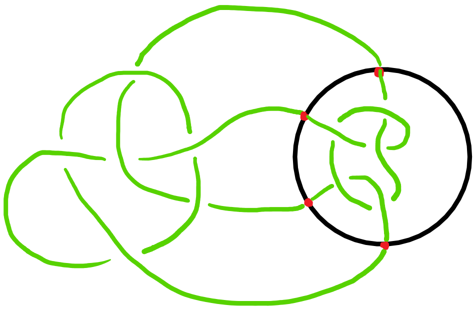
$\not\Downarrow$ $\Uparrow$

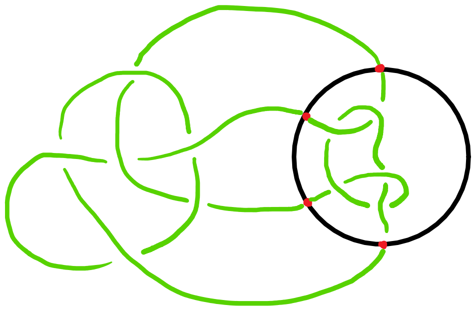
- K is called (smoothly) <u>slice</u> if $D^2 \subset B^4$ smoothly embedded

e.g. $K \# \overline{K}$



e.g. Conway knot   Kinoshita–Terasaka knot





	Conway knot	Kinoshita–Terasaka knot
top. slice	✓	✓
slice	?	✓

Problem: KT knot is slice.
+ve mutant of Conway knot.

Solution: Get as far from KT knot as possible!

§2 Handles and Kirby calculus

— Handles

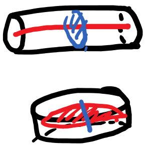
Def. An n-dim'l k-handle is $H^n_k \cong \underset{\text{core}}{D^k} \times \underset{\text{cocore}}{D^{n-k}}$, glued in a particular way.

index

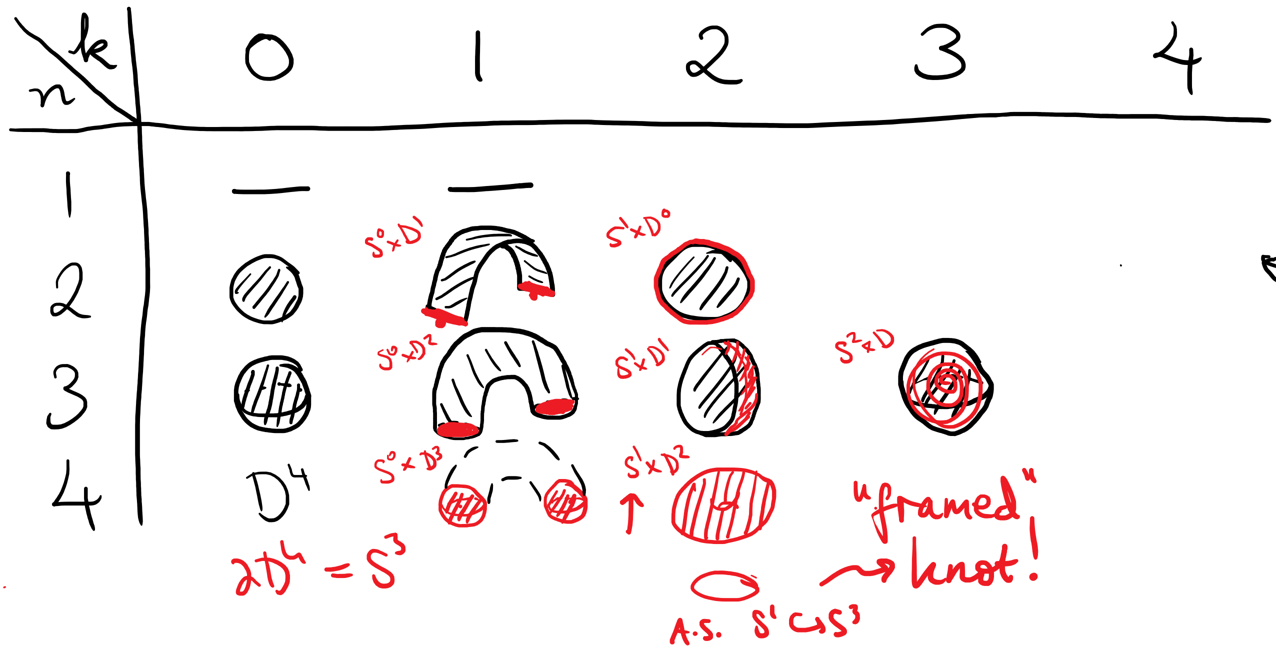
$$\partial H^n_k \cong (\partial D^k \times D^{n-k}) \cup (D^k \times \partial D^{n-k})$$

$$\cong \underbrace{(S^{k-1} \times D^{n-k})}_{\text{attaching region}} \cup \underbrace{(D^k \times S^{n-k-1})}_{\text{belt region}}$$

attaching sphere.

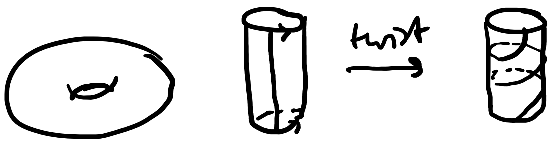
H^3_1 

H^3_2



"Framing"

$r \in \mathbb{Z}$

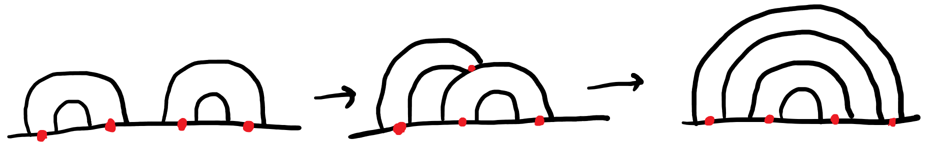


— Kirby calculus

Thm. [Kirby] L, L' framed link diagrams related by a finite sequence of "Kirby moves"
$\Rightarrow$ corresponding 4-manifolds X, X' diffeomorphic

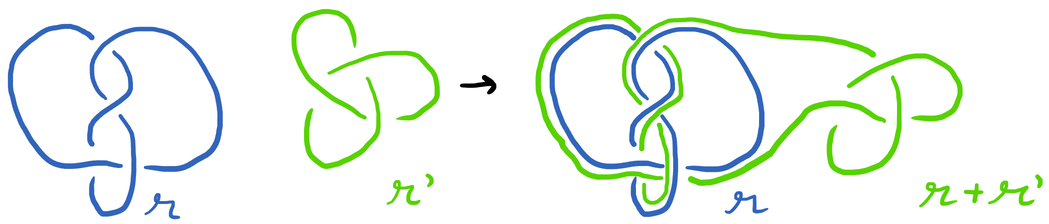
· Handle slides

2D



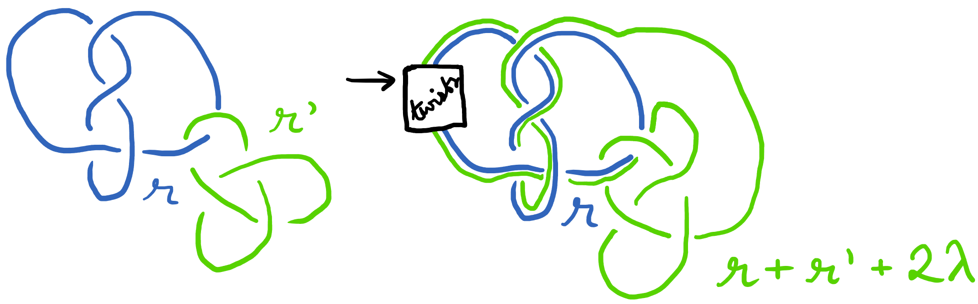
3D

4D

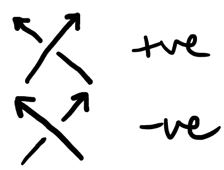


linking number.

$\downarrow$

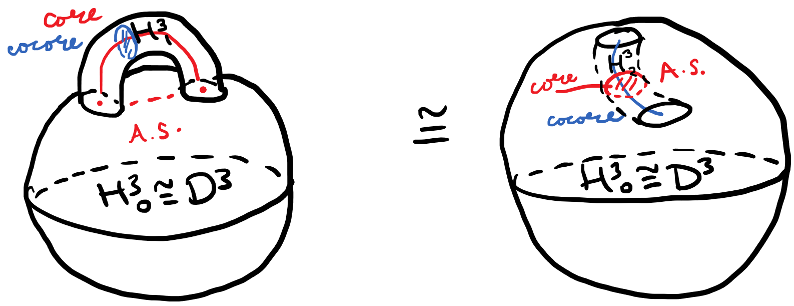


$$\lambda = lk(K, K')$$
$$= \frac{1}{2}\Big(\#\{+ve \text{ crossings between } K, K'\} - \#\{-ve \text{ crossings between } K, K'\}\Big)$$

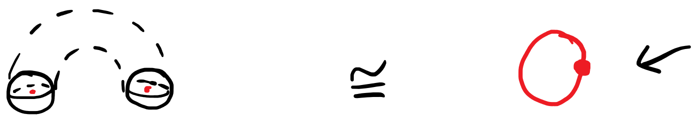


- Attaching a 1-handle = removing a 2-handle

3D



4D



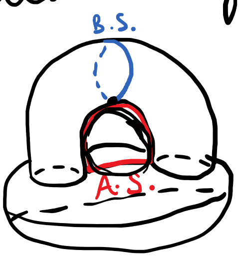
dotted circle notation

$H^4_1 \cong D^1 \times D^3$ $\qquad$ $H^4_2 \cong D^2 \times D^2$

A.S. $= S^0$ $\qquad$ A.S. $= S^1$

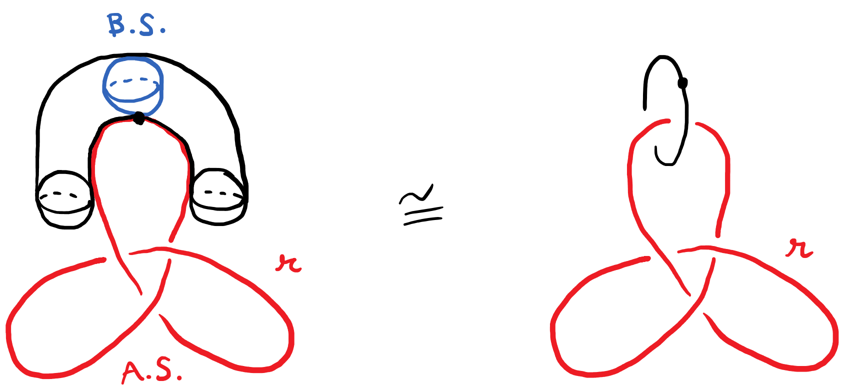
- Cancellation of handle pairs

3D



$$|\text{A.S.}(H^3_2) \pitchfork \text{B.S.}(H^3_1)| = 1$$

4D



cancel (delete from diagram)

§3 Construction of Piccirillo's knot

– Trace

Def. The trace of K is the 4-manifold $X(K) \cong B^4 \cup_K H^4_2$ where H^4_2 is glued to B^4 along K with framing 0.

Lemma K is slice $\Leftrightarrow$ $X(K) \hookrightarrow S^4$ smoothly.
[Kirby & Melvin].

Corollary $X(K) \cong X(K') \Rightarrow (K \text{ is slice} \Leftrightarrow K' \text{ is slice})$

Strategy: Construct K' s.t. $X(K) \cong X(K')$, where K is the Conway knot. Then show K' is not slice.

– Dualisable patterns construction

Step 1 Given a link $L = B \cup R \cup G$, we'll construct two knots K, K' which satisfy $X(K) \cong X(K')$.

L must satisfy 3 conditions:

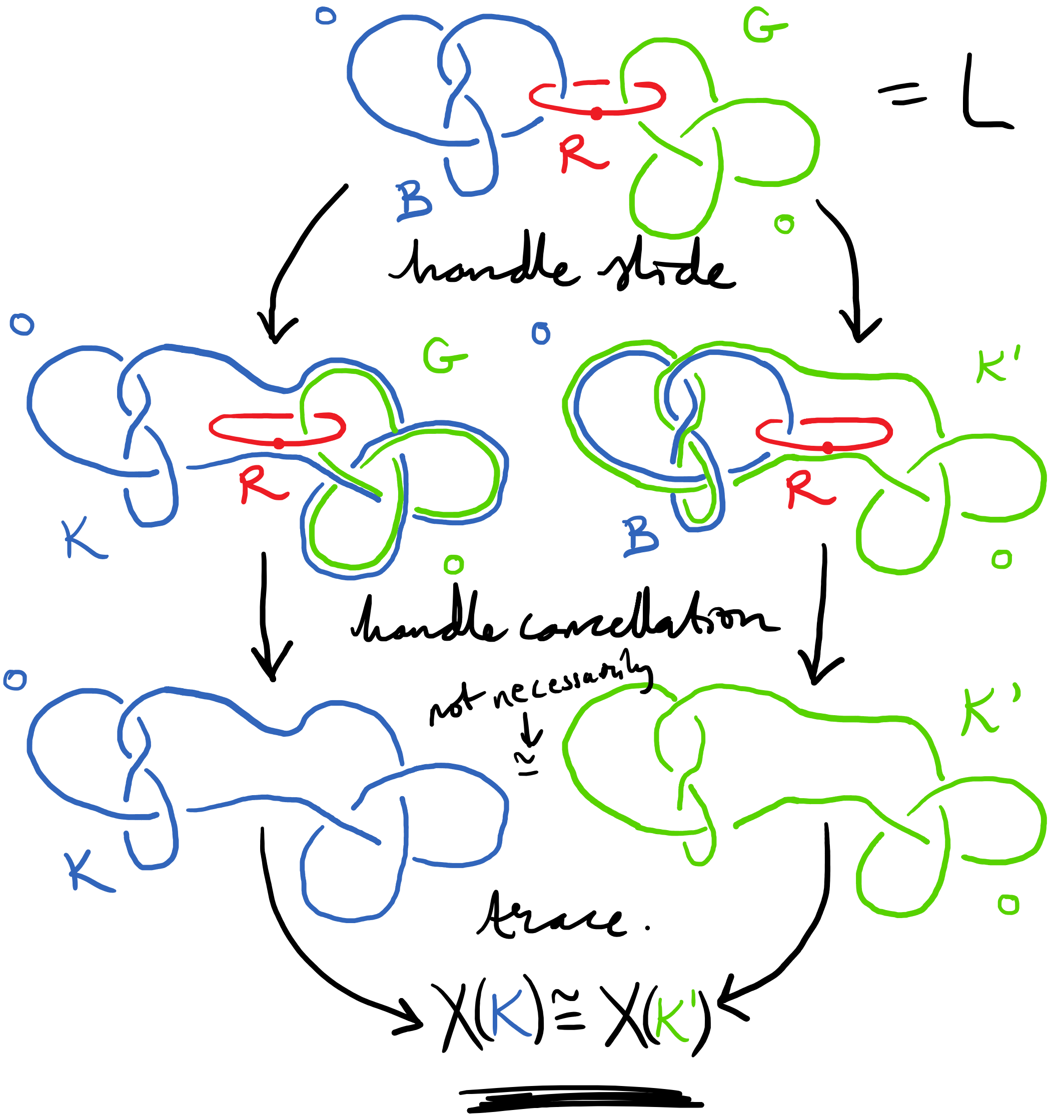
(i) $B \cup R \simeq B \cup \mu_B$

(ii) $G \cup R \simeq G \cup \mu_G$

(iii) $lk(B, G) = 0$

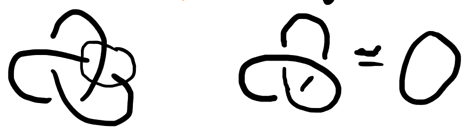
μ = meridian

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & K' \arrow[dl] \\ & X(K) \cong X(K') & \end{tikzcd}$$

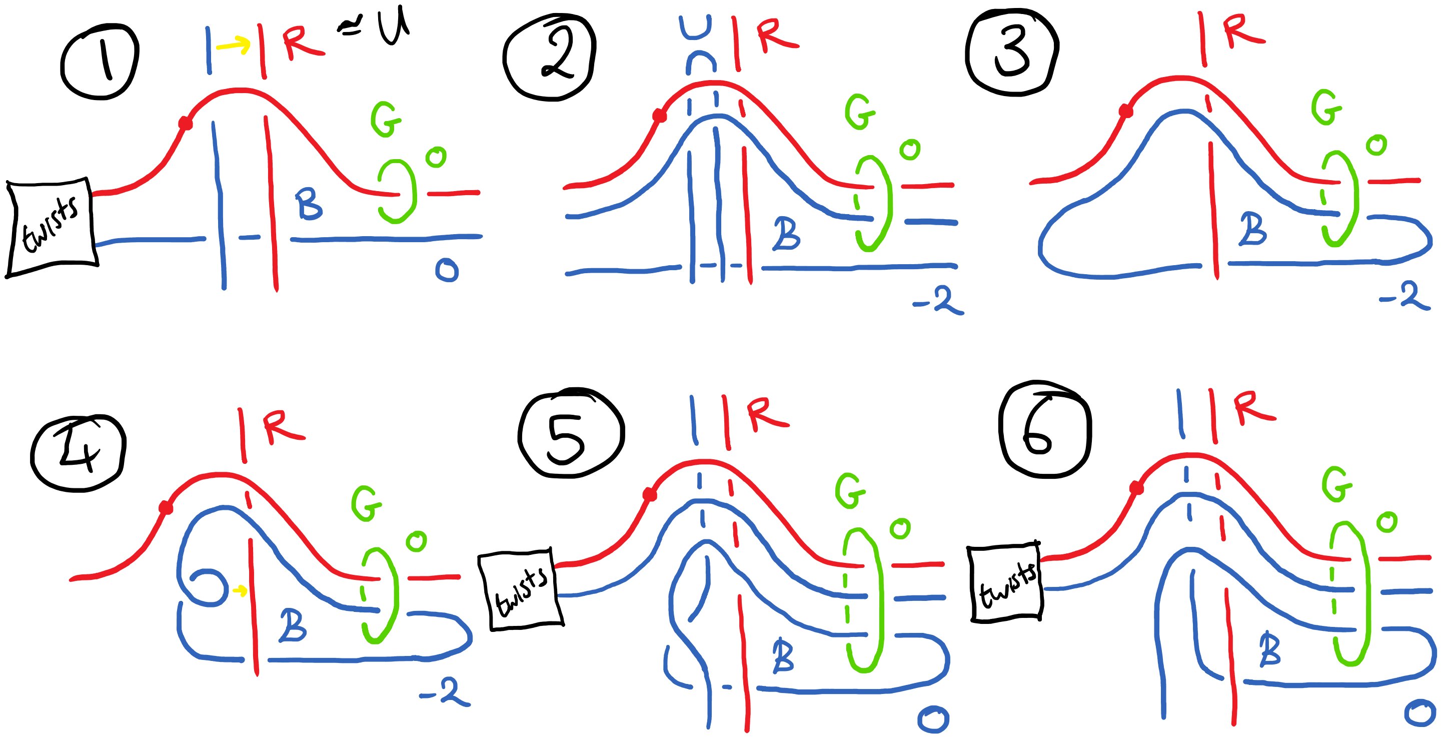


Step 2 Let K be the Conway knot
Go backwards to find L.
Then construct K'.

Prop. If K has unknotting number 1, "trefoil" then such an L exists.

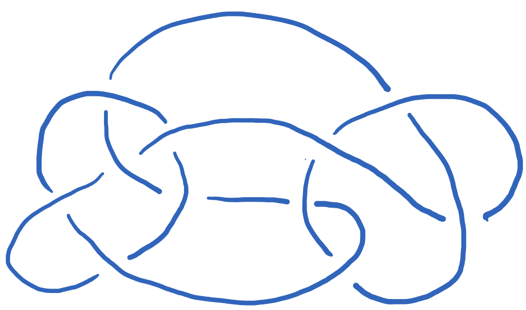


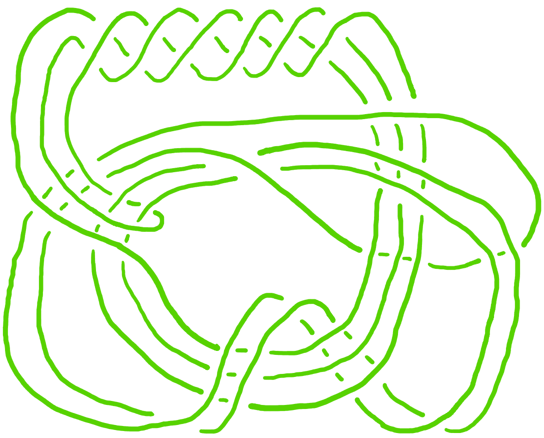
Proof:



K = Conway knot

K' = Piccirillo's knot

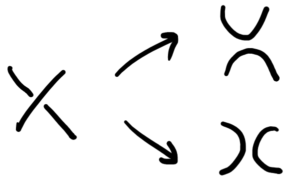




$$X(K) \cong X(K')$$

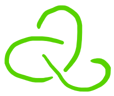
§4 Rasmussen's s-invariant

– Khovanov homology

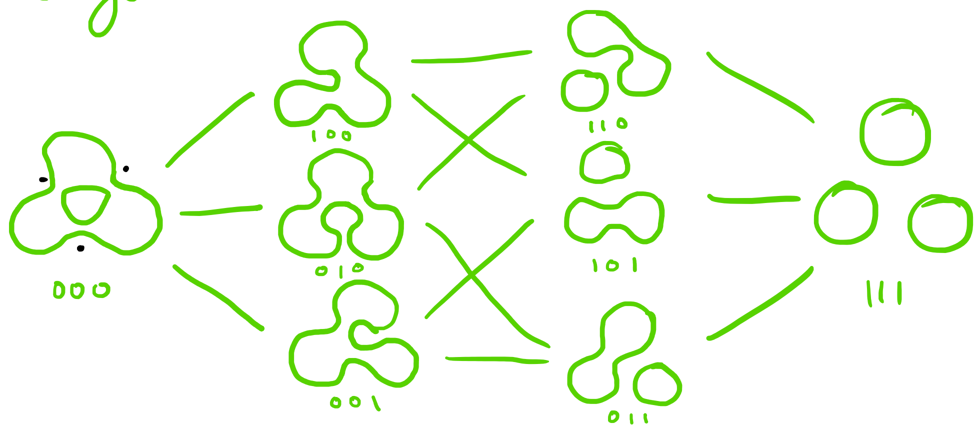
L link. "Resolve" crossings. 

$\downarrow$

n crossings $\longrightarrow$ 2^n resolutions

e.g. K = trefoil  "cube of resolutions"

n=3



TQFT

$\bigcirc \longrightarrow V$ $\qquad$ $\bigcirc \xrightarrow{\text{split}} \bigcirc\bigcirc \longrightarrow \Delta : V \to V \otimes V$

$\bigcirc\bigcirc \longrightarrow V \otimes V$ $\qquad$ $\bigcirc\bigcirc \xrightarrow{\text{merge}} \bigcirc \longrightarrow m : V \otimes V \to V$

i = homological grading

j = quantum grading. $\qquad$ bigraded

$\downarrow$

$\rightsquigarrow$ chain complex $\qquad CKh^{i,j}(L) \qquad E^1$

$\rightsquigarrow$ Khovanov homology $\qquad Kh^{i,j}(L) \qquad E^2$

$\vdots \qquad \vdots$

$\rightsquigarrow$ Lee homology $\qquad KhL^{i,j}(L) \qquad E^\infty$

$$KhL(K) \cong \mathbb{Q} \oplus \mathbb{Q}$$

- s-invariant

Theorem [Rasmussen]

For any knot K, the generators of $KhL(K)$ are located in the gradings $(i, j) = (0, s(K) \pm 1)$.

If K is slice, then $s(K) = 0$.

s-invariant

So NTS $s(K') \neq 0$.

Compute $Kh(K')$.

$j \backslash i$	-3	-2	-1	$\boxed{0}$	1	2	3	4	$\cdots$
$\vdots$									
5					1	3	3	2	
3				1	3	3			
1			2	2	2				
-1		1		1					
-3		2							
-5	1								

$s(K') = 2$

$s(K') = 0$

$(i, j) = (0, 3)$

$\underline{s(K') = 2} \neq 0 \Rightarrow K'$ not slice

$\Rightarrow K$ not slice.