Hyper-kähler Geometry

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"Hyperkähler manifolds"

§0 Introduction

§0.1 Quaternions: the complex numbers of the complex numbers

$$\mathbb{H} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$$

$i^2 = j^2 = k^2 = ijk = -1$ ($\Rightarrow ij = -ji = k$ and same for cycl. perm)
(real #'s comm.)

$$\mathbb{H} = \{z + wj \mid z, w \in \mathbb{C}\} \qquad z = a+bi, \; w = c+di$$

$j^2 = -1$ and $ij = -ji$
(real #'s comm)

We can conjugate $a+bi+cj+dk \mapsto a-bi-cj-dk$

$$z + wj \mapsto \bar{z} - wj \qquad \mathbb{H} \circlearrowleft V$$

A v.s. V is (left/right) quatern. if it has a ——"—— repr of $\mathbb{H}$

If f.dim $\leadsto \dim_{\mathbb{R}} V = 4m$ and it will be isom. $\mathbb{H}^m$. $q\cdot(q_1,\dots,q_n) = (qq_1,\dots,qq_n)$

We can have Herm inner prods

Note: A cx vector space will be quat. if it has an $J \in End(V)$ which is conj-lin and $J^2 = -1$

$$ij = -ji \qquad j^2 = -1$$
$$i^2 = -1$$
$$k := ij$$
$$k^2 = -1 \quad \ldots$$

A quat. v.s. is, in part, complex. E.g. mult by $i \in \mathbb{H}$ will be lik mult by $i \in \mathbb{C}$

Also —— " —— $j \in \mathbb{H}$ —— " ——

$$k \in \mathbb{H}$$

$$ai + bj + ck \in \mathbb{H}$$

if $a^2 + b^2 + c^2 = 1$.

we have an S^2 of cx str's.

§0.2 Hyper-Kähler (h-k) manifolds:

the Kähler mflds of Kähler mflds?

<u>Def</u>: $(M, g, \omega_1, \omega_2, \omega_3, I_1, I_2, I_3)$ is h-k if:

- (M, g) is Riemannian
- (M, ω_i) is sympl. $\forall i$
- (M, I_i) is cx. $\forall i$

and

$$g(X, Y) = \omega_i(X, I_i Y) \quad \forall i$$

$$I_i^2 = I_1 I_2 I_3 = -1 \quad \forall i$$

Note: (M, g, ω_i, I_i) is Kähler

in fact, $(M, g, \sum a_i \omega_i, \sum a_i I_i)$ is Kähler if $\sum a_i^2 = 1$

§1 Complex/Kähler viewpoint

We fix one of the Kähler str's $(M^{4m}, g, \omega_1, I_1)$. Then, we can write $\Omega^{\bullet} = \bigoplus_{0 \leq p,q \leq 2m} \Omega^{p,q}$ $(\dim_{\mathbb{C}} M = 2m)$

We know:

- $\omega_1 \in \Omega^{1,1}$
- $d\omega_1 = 0$
- $\omega_1^{2m} \in \Omega^{2m,2m} = \Omega^{4m}$ is non-vanishing (a volume form)

Now, consider $\omega_c = \omega_2 + i\omega_3$. It turns out:

- $\omega_c \in \Omega^{2,0}$
- $d\omega_c = 0 \Rightarrow \partial\omega_c = 0$
- $\omega_c^m \in \Omega^{2m,0}$ is non-vanishing (hol. vol. form) $\Rightarrow$ CY

So ω_c is a hol sympl. form on (M^g, ω_1, I_1)

In fact, if M is closed, $\exists$ hol. sympl. form $\Leftrightarrow$ h-K.

§2 Riemannian viewpoint

Def: If (M^{4m}, g) is Riem., it's h-K if
$$\mathrm{Hol}(g) \subseteq Sp(m) \quad (\subseteq O(4m))$$

Recall: holonomy gives a str on TM which is cov. const wrt the LC conn. of g

$$Sp(m) = U(m, \mathbb{H}) \quad \text{(preserve a Herm. i.p. on } \mathbb{H}^m\text{)}$$
$$= GL(m, \mathbb{H}) \cap O(4m, \mathbb{R})$$
$$= Sp(2m, \mathbb{C}) \cap U(2m, \mathbb{C}) \quad \text{(cf. w/ hol sympl)}$$

cf $U(n) = U(n, \mathbb{C})$
$$= GL(n, \mathbb{C}) \cap O(4m, \mathbb{R})$$
$$= Sp(2n, \mathbb{R}) \cap O(2n, \mathbb{R})$$

We have I_1, I_2, I_3 s.t.
$$I_i^2 = I_1 I_2 I_3 = -1$$
$$\nabla^{LC} I_i = 0$$

Also, we note that

$$Sp(m) \subseteq SU(2m)$$

so h-K $\Rightarrow$ CY $\omega_c^m = \Theta$

If $m=1$ $Sp(1) = SU(2)$ $(= S^3)$

so h-K $\Leftrightarrow$ CY for $4_{\mathbb{R}}$-mfld

$\Theta = \omega_c$

§3 Symplectic viewpoint

If $G \circlearrowright (M, \omega)$ we might have

$\mu : M \to \mathfrak{g}^*$ G-equiv.

s.t. $\forall X \in \mathfrak{g}$ $d\langle\mu, X\rangle = \underbrace{\iota_{X^*}\omega}$

This is a Hamiltonian action

we then have $G \circlearrowright \mu^{-1}(0)$. Under some conditions, $\mu^{-1}(0)/G$ is a manifold, and it inherits a sympl str

$$M/\!/G := \mu^{-1}(0)/G \quad (\dim M/\!/G = \dim M - 2\dim G)$$

Furthermore, if (M, g, ω, I), and G is hol.

then $M/\!/G$ is also Kähler

Consider $G^{\mathbb{C}} \circlearrowright (M, g, \omega, I)$ $\qquad (S^1)^{\mathbb{C}} = \mathbb{C}^*$

Then, $M/\!/G = M^s / G^{\mathbb{C}}$ (M^s pts of M whose $G^{\mathbb{C}}$-orbit inters. $\mu^{-1}(0)$)

Suppose $G \circlearrowright (M, g, \omega_i, I_i)$, we might get $\mu_1, \mu_2, \mu_3 : M \to \mathfrak{g}^*$

We can write them $\mu : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$

Then, we can try to do:

$M /\!/\!/ G = \mu^{-1}(0)/G$. This is h-k!

Note: $\dim M /\!/\!/ G = \dim M - 4 \dim G$

Fix one of the kähler str's (M, g, ω_1, I_1).

Then, consider: $\mu_1 : M \to \mathfrak{g}^*$ $\quad (\omega_1)$

$$\mu_{\mathbb{C}} = \mu_2 + i\mu_3 : M \longrightarrow (\mathfrak{g}^{\mathbb{C}})^* \quad (\omega_{\mathbb{C}} = \omega_2 + i\omega_3)$$

$$M /\!/\!/ G = \mu^{-1}(0)/G = \mu_{\mathbb{C}}^{-1}(0) \cap \mu_1^{-1}(0) / G$$

$$= \mu_{\mathbb{C}}^{-1}(0) /\!/ G = \mu_{\mathbb{C}}^{-1}(0) / G^{\mathbb{C}} \quad \text{wrt } I_1$$

$$\text{``} = M /\!/ G^{\mathbb{C}} \quad \text{wrt } I_1, \omega_{\mathbb{C}} \text{''}$$

Suppose that $(\mu_c^{-1}(0), g, \omega_1, I_1)$

$G^{\mathbb{C}} \circlearrowright \mu_c^{-1}(0)$, $G^{\mathbb{C}} \circlearrowright (M, g, \omega_1, I_1, \omega_c)$, $\omega_c \rightsquigarrow \mu_c$

§4 Examples

① $S^1 \circlearrowright M = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$

$$M /\!\!/\!\!/_{S^1} = T^* \mathbb{CP}^{n-1}$$

(for $n=2$, this Eguchi-Hanson)

② suppose M^4 is h-K and $\begin{array}{c} P \\ \downarrow \\ M \end{array}$

Consider $\mathcal{A} = \{\text{conn. on } P\}$

this is affine over $\Omega^1(\operatorname{ad} P)$

this is a quat. v. s.

So $\mathcal{A}$ is h-K

Consider $\mathcal{G} \circlearrowright \mathcal{A}$. It preserves the h-k.

$$\mu : \mathcal{A} \longrightarrow (\Gamma(\operatorname{ad} P))^* \times \mathbb{R}^3 \qquad \Omega^+ = \langle \omega_1, \omega_2, \omega_3 \rangle$$

$$= \Omega^+(\operatorname{ad} P)$$

$$A \longmapsto F_A^+ \qquad \text{(we need additional conds)}$$

So $\mathcal{A}/\!/\!/G$ = moduli of ASD conn mod gauge trans

which might be h-K

h-K $\Rightarrow$ Ricci-flat

dim = 4 : $\mathbb{T}^4$, K3-surf

higher dim: Hilbert schemes of points on $\mathbb{T}^4$, $K3$

H.s. of k pts on M is $\sim M \times M \times \cdots \times M / \Sigma_k$