Barrier methods for minimal submanifolds and the Gibbons–Hawking ansatz

Federico Trinca

University of Oxford

January 21, 2021

Overview

Mean curvature and second fundamental form

Let (M^n, g) be a Riemannian manifold and let Σ^k be a submanifold of M with the induced Riemannian structure.

Definition

Let ∇ be the Levi-Civita connection of (M, g). The second fundamental form of Σ is:

$$A(X,Y) := (\nabla_X Y)^{\perp}$$
$$II(X,Y) := g(A(X,Y),\nu),$$

where X, Y are tangent vectors and ν is a normal vector.
The mean curvature H of Σ is the trace of A i.e.

$$H = \sum_i A(e_i, e_i), \quad \{e_i\}_i \text{ local orthonormal frame of } \Sigma.$$

Minimal submanifolds

Definition

A submanifold Σ of a Riemannian manifold is minimal if it is a critical point of the volume. By the first variation formula, Σ is minimal if and only if $H = 0$.

Example

1. Geodesics are 1-dimensional minimal submanifolds;
2. Plane, catenoid, Enneper surface in $\mathbb{R}^3$;
3. The clifford torus in S^3;
4. Complex submanifolds of Kähler manifolds;
5. Calibrated submanifolds are homologically volume minimizing and hence minimal.

k-convex functions

Definition

A smooth function $f : M^n \to \mathbb{R}$ is said to be k-convex if

$$\mathrm{Tr}_W \mathrm{Hess} f_x \geq 0 \quad \forall x \in M, \forall W \in G(k, T_x M).$$

If the inequality is strict, f is strictly k-convex.

We recall the following well-known lemma.

Lemma

Let $f : M^n \to \mathbb{R}$ be a k-convex function and let Σ^k be a k-dimensional compact minimal submanifold. Then, Σ is contained in the set where f is not strict. In particular, $f|_\Sigma$ is constant.

Proof: $Tr_\Sigma \mathrm{Hess} f = \Delta_\Sigma f - \mathrm{H}(f)$. □

Examples

- In $\mathbb{R}^n$ with the Euclidean metric, $f(x) = |x|^2$ is 1-convex.
- In $\mathbb{R}^4$ with Taub–NUT metric, $f(x) = |x|^2$ is 1-convex.
- (Tsai–Wang 2018) In T^*S^2 with Eguchi–Hanson metric, the square of the distance from the zero section is 1-convex.
- (Tsai–Wang 2018) In T^*S^n ($T^*\mathbb{CP}^n$) with Stenzel metric (Calabi metric), the square of the distance from the zero section is 1-convex.
- (Tsai–Wang 2018) In $\mathbb{S}(S^3)$, $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{CP}^2)$ and $\mathbb{S}_-(S^4)$ with the Bryant–Salamon metrics, the square of the distance from the zero section is 1-convex.

In particular, compact minimal submanifolds are contained in the zero section (minimal).

k-convex boundaries

Let Ω be a domain of M^n.

Definition

We say that $\partial\Omega$ is k-convex if

$$\mathrm{Tr}_W II_x \geq 0 \quad \forall x \in \partial\Omega, \forall W \in G(k, T_x\partial\Omega),$$

where II is the second fundamental form with respect to the inward pointing normal. If the inequality is strict, $\partial\Omega$ is strictly k-convex.

Theorem (Harvey–Lawson 2012)

If $\partial\Omega$ is strictly k-convex, there is a k-convex function $f \in C^\infty(\overline{\Omega})$ which is strict in a neighbourhood of $\partial\Omega$.

The barrier method

Corollary

If $\partial\Omega$ is strictly k-convex, there are no k-dimensional compact minimal submanifolds contained in Ω with a point tangent to $\partial\Omega$.

Remark

$n-1$ convex $\iff$ inward pointing mean curvature.

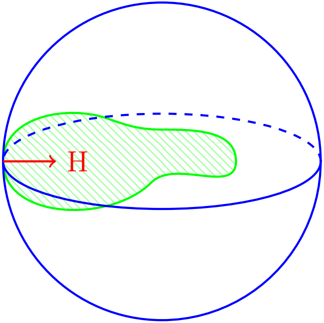
Remark

Let $f : M \to \mathbb{R}$ and let a be a regular value. Then, the second fundamental form of $f^{-1}(a)$ is:

$$II = \frac{1}{|\nabla f|}\mathrm{Hess} f$$

Avoidance principle

If $k = n - 1$, it is just the classical avoidance principle for the mean curvature flow. In higher codimension, we can use the generalized avoidance principle (White '15).



The Gibbons-Hawking ansatz

Let $U \subset \mathbb{R}^3$ open, let $\pi : X \to U$ be a principal S^1-bundle, let ξ generator of the action and let $\eta \in \Omega^1(X, \mathbb{R})$ connection 1-form i.e. S^1-invariant and $\eta(\xi) = 1$.
Let ϕ be a positive harmonic function on U satisfying:

$$*_{\mathbb{R}^3} \mathrm{d}\phi = \mathrm{d}\eta \qquad \text{(Monopole equation)}.$$

Then, (X, g) is an hyperkähler manifold constructed via the Gibbons-Hawking ansatz,

$$g := \phi g_{\mathbb{R}^3} + \phi^{-1}\eta^2,$$
$$\omega_i := \mathrm{d}x_i \wedge \eta + \phi \mathrm{d}x_j \wedge \mathrm{d}x_k.$$

Examples

- $\phi = \frac{1}{2|x|} \implies$ Euclidean space.
- $\phi = m + \frac{1}{2|x|} \implies$ Taub–NUT space.
- $\phi = \frac{1}{2|x-p|} + \frac{1}{2|x+p|} \implies$ Eguchi–Hanson space.
- $\phi = \sum_{i=1}^{k} \frac{1}{2|x-p_i|} \implies$ Multi-Eguchi–Hanson space.
- $\phi = m + \sum_{i=1}^{k} \frac{1}{2|x-p_i|} \implies$ Multi-Taub–NUT space.

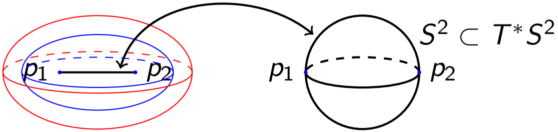


Figure: Equivalence of E–H metric to two center G–H metric.

Circle-invariant minimal submanifolds

Let (X, g) multi-E-H or a multi-T-N space with k singular points denoted by $\{p_i\}_{i=1}^k$.
Using Hsiang and Lawson equivariant argument we have:

- (Lotay–Oliveira 2020) S^1-invariant geodesics in $(X, g) \iff \nabla\phi = 0$. There are $k-1$ (unstable) S^1-invariant geodesics and are contained in $\mathrm{Co}(\{p_i\}_i)$.
- (Lotay–Oliveira 2020) S^1-invariant minimal surfaces in $(X, g) \iff$ geodesics in Euclidean $\mathbb{R}^3$. These are complex curves w.r.t a compatible complex structure and contain the class of all compact complex curves (segment connecting singular points).
- (T. 2020) S^1-invariant minimal hypersurfaces in $(X, g) \iff$ minimal surfaces in $(\mathbb{R}^3, \phi^{1/2} g_{\mathbb{R}^3})$. Only known examples are given by symmetries of the "singular points".

Motivation

Question

Are compact minimal submanifolds S^1-invariant or contained in a S^1-invariant submanifold?

Remark

- In the Euclidean case and in the Taub–NUT case, it vacously holds.
- Tsai and Wang proved it in the E-H case.
- Compactness is crucial

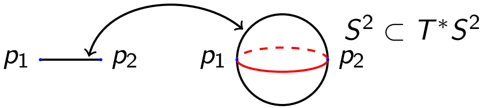
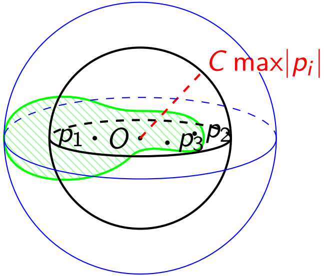


Figure: Karigiannis-Min-Oo construction not circle-invariant.

Spherical barriers I

Lemma (T. 2020)

The S^1-invariant hypersurface in X corresponding to the Euclidean sphere S_r is strictly 3-convex w.r.t the interior of the sphere for all $r > 4/3 \max_i |p_i|_{\mathbb{R}^3}$ and all $r < \min\{|p_i|_{\mathbb{R}^3} : |p_i|_{\mathbb{R}^3} > 0\}$. Moreover, it is strictly 1-convex if $r > C \max_i |p_i|_{\mathbb{R}^3}$, where $C \approx 5.07$ and for r small enough when centered in a p_i.



Spherical barriers II

Theorem (T. 2020)

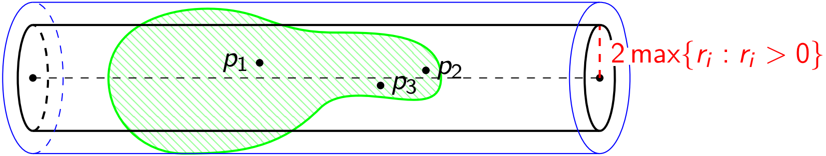
Compact minimal hypersurfaces (submanifolds) need to be contained in $\pi^{-1}(\{|x|_{\mathbb{R}^3} \leq 4/3(C)\max_i |p_i|_{\mathbb{R}^3}\})$. Moreover, there are no compact minimal hypersurfaces contained in $\pi^{-1}(\{|x|_{\mathbb{R}^3} < \min\{|p_i|_{\mathbb{R}^3} : |p_i|_{\mathbb{R}^3} > 0\}\})$.

Idea of the proof: Relate IIFF of the hypersurface in X to the IIFF of the projecting surface in $\mathbb{R}^3$ plus terms involving ϕ and $\nabla_{\mathbb{R}^3}\phi$. Diagonalize the second fundamental form of the surface we obtain a 3×3 matrix which is simple enough to study its convexity. Harvey and Lawson barriers let us conclude. □

Cylindrical barriers I

Lemma (T. 2020)

The S^1-invariant hypersurface in X corresponding to the Euclidean Cylinder $\Sigma_r := \{x_1^2 + x_2^2 = r^2\}$ is strictly 3-convex w.r.t the interior of the cylinder for all $r > 2\max_i r_i$ and all $r < \min\{r_i : r_i > 0\}$, where $r_i := \sqrt{(p_i)_1^2 + (p_i)_2^2}$.



Cylindrical barriers II

Theorem (T. 2020)

Compact minimal hypersurfaces need to be contained in $\pi^{-1}(\{|x|_{\mathbb{R}^3} \leq 2\max_i r_i\})$. Moreover, there are no compact minimal hypersurfaces contained in $\pi^{-1}(\{|x|_{\mathbb{R}^3} < \min\{r_i : r_i > 0\}\})$.

Corollary (T. 2020)

There are no compact minimal hypersurfaces in the collinear case.

Idea of the proof: Analogous to the spherical case □

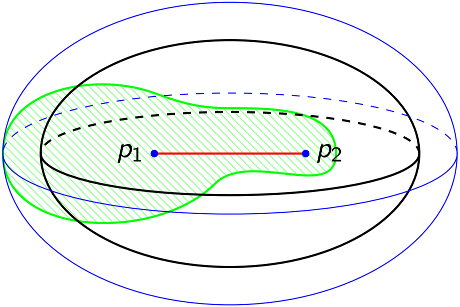
Remark

Differently from the spherical case, hypersurfaces corresponding to Euclidean cylinders cannot be 1 or 2 convex.

Ellipsoidal Barrier I

Lemma (T. 2020)

In the two point case, the S^1-invariant hypersurface corresponding to the Euclidean ellipsoid Σ_r is strictly 1-convex with respect to the interior of the ellipsoid for all $r > 0$.



Ellipsoidal Barrier II

Theorem (T. 2020)

In the two point case compact minimal submanifolds are contained in the unique S^1-invariant compact minimal surface.

Corollary

If we have *at most* two singular points, compact minimal submanifolds are S^1-invariant, or are contained in one.

Remark

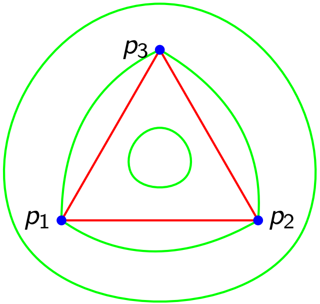
In particular, we can reckon our theorems as extensions to the multi-point case of the classical barrier theorem for the Euclidean (Taub-NUT) space and of Tsai and Wang barrier theorem for the Eguchi-Hanson space.

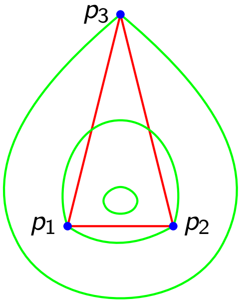
k-ellipsoidal barriers?

1 point $\Longrightarrow$ spheres are convex

2 points $\Longrightarrow$ ellipsoids are convex

k points $\overset{?}{\Longrightarrow}$ k-ellipsoids are convex





Local barriers

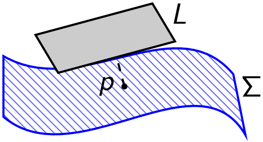
WLOG: we can only consider compact complex surfaces.

Proposition (Tsai and Wang 2018)

Given a compact minimal surface with everywhere positive Gaussian curvature, there exists a neighbourhood in which the square of the distance function is 2-convex.

Proposition (T. 2020)

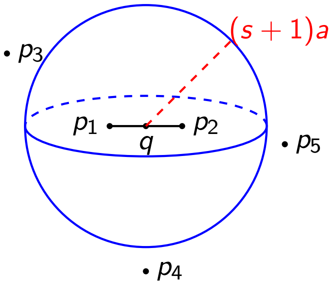
Given a compact minimal surface with a point of negative Gaussian curvature, every neighborhood of the surface admits a point where the square of the distance function is not 2-convex.



Existence local barriers

Proposition (Lotay-Oliveira 2020, T. 2020)

If $\{p_i\}_{i=3}^k$ are sufficiently distant, w.r.t the Euclidean metric, from the midpoint of p_1 and p_2, then the S^1-invariant minimal surface corresponding to the segment $\overline{p_1 p_2}$ has everywhere positive Gaussian curvature.



Non-existence local barriers

Proposition (T. 2020)

If $p_1 = (0, 0, 1)$, $p_2 = (0, 0, -1)$ and $p_3 = (0, \epsilon, 0)$ are the singular points, then there exists an ϵ small enough such that the Gaussian curvature is negative at $\pi^{-1}(0)$.

Conclusion

Hence, we have shown that the natural barriers are not strong enough, not even locally, to prove that compact minimal submanifolds are circle-invariant or contained in one for a generic multi-Eguchi–Hanson or multi-Taub–NUT space.

Thank You!