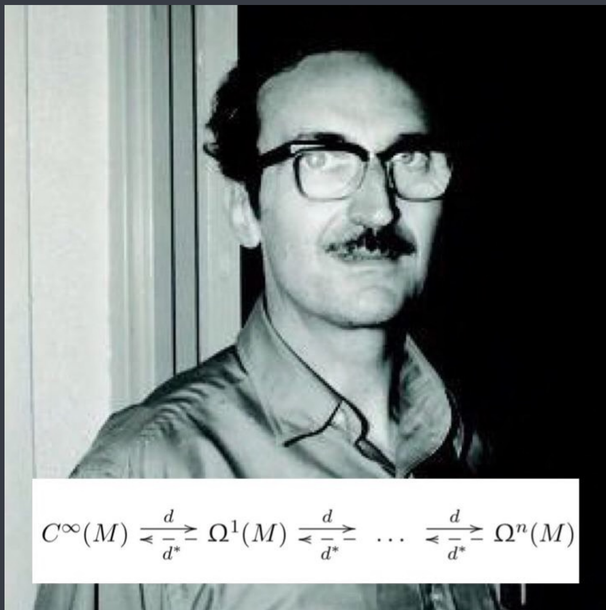
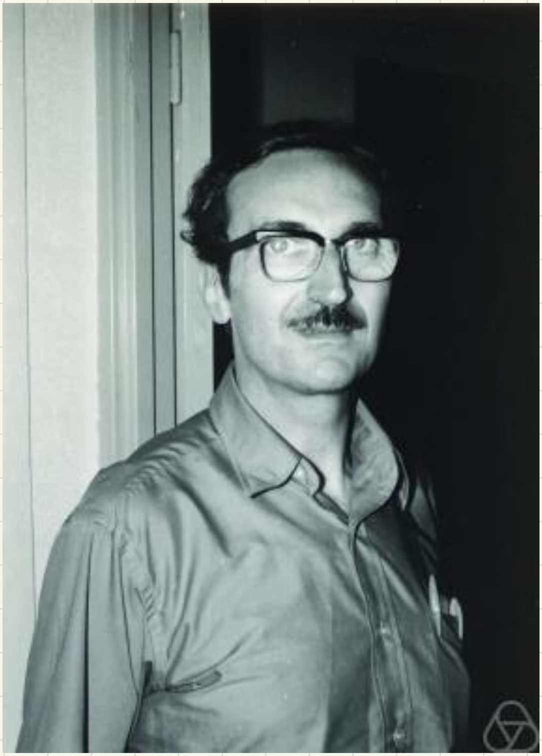



# Intrinsically harmonic forms

# Advertisement

Global Analysis, Papers in Honour of K. Kodaira, Princeton University Press (1969)

- Artin
- Atiyah
- Calabi
- Chern
- de Rham
- Grauert
- Griffiths
- Hirzebruch
- Mumford
- Satake
- Thom
- Nirenberg



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Joaquin Phoenix ('Her')

## (1) Prerequisites: (Briefly) Hodge theory on Riemannian manifolds

Let $V$ be an $n$-dim. real vector space. Pick an orientation and inner product $g = \langle - | - \rangle$ on $V$.

- **Volume form**: $\omega_g = e^1 \wedge \ldots \wedge e^n \in \Lambda^n V^*$ for any oriented orthonormal basis $\Rightarrow$
$\Rightarrow \Lambda^n V^* \simeq \mathbb{R}$ $(\omega_g \longleftrightarrow 1)$;

- $\Lambda^k V^* \times \Lambda^{n-k} V^* \xrightarrow{\wedge} \Lambda^n V^* \simeq \mathbb{R}$ nondegenerate pairing $\Rightarrow$
$\Rightarrow \Lambda^k V \simeq \Lambda^{n-k} V^*$. But we have $V \simeq V^* \Rightarrow \Lambda^k V \simeq \Lambda^k V^* \Rightarrow$
$\Rightarrow \Lambda^k V^* \underset{*}{\xrightarrow{\simeq}} \Lambda^{n-k} V^*$

the Hodge star operator

- If $e_1, \ldots, e_n$ is an oriented orthonormal basis for $V$, then:

$$*: e^{i_1}\wedge\ldots\wedge e^{i_k} \mapsto \operatorname{sgn}(i_1,\ldots,i_k,j_1,\ldots,j_{n-k})\cdot e^{j_1}\wedge\ldots\wedge e^{j_{n-k}},$$

where $1\leq i_1<\ldots<i_k\leq n$, $1\leq j_1<\ldots<j_{n-k}\leq n$, and $\{i_1,\ldots,i_k\}\sqcup\{j_1,\ldots,j_{n-k}\}=\{1,\ldots,n\}$.

**Example**: 
$$\begin{tikzcd} \mathbb{R}\cong\Lambda^0V^* \arrow[r, shift left, "*"] & \Lambda^nV^* \arrow[l, shift left, "*"] , \\[-20pt] \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\[-20pt] 1 \arrow[r, leftrightarrow] & \omega_g \end{tikzcd}$$

Riem. metric $g$

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If $M$ is an oriented Riem. $n$-manifold, the same can be done pointwisely for each $T_pM$.

- We obtain vector bundle isom-m:

$$*: \Lambda^kT^*M \xrightarrow{\sim} \Lambda^{n-k}T^*M$$

on sections

$$*: \Omega^k(M) \xrightarrow{\sim} \Omega^{n-k}(M)$$

**Remark**: We write $\omega_g\in\Omega^n(M)$ for the volume form.

- The Riem. metric $g$ can be extended to various tensor bundles. E.g.: $\omega,\eta\in\Omega^k(M)\Rightarrow\langle\omega|\eta\rangle_g\in C^\infty(M)$; if, locally, $\omega=\omega^1\wedge\ldots\wedge\omega^k$, $\eta=\eta^1\wedge\ldots\wedge\eta^k$, then

$$\langle \omega | \eta \rangle_g = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \langle \omega^1 | \eta^{\sigma(1)} \rangle \cdot \ldots \cdot \langle \omega^k | \eta^{\sigma(k)} \rangle = \det \langle \omega^i | \eta^j \rangle_{ij}$$

**Properties:** (1) $*^2 = (-1)^{k(n-k)}$;

(2) $f \in \Omega^0(M) = C^\infty(M) \Rightarrow *f = f\omega_g,\ *(f\omega_g) = f$;

(3) $\omega, \eta \in \Omega^k(M) \Rightarrow \omega \wedge *\eta = \langle \omega | \eta \rangle_g \cdot \omega_g$.

- Next, define a codifferential:

$$\Omega^{k-1}(M) \underset{d^*}{\overset{d}{\rightleftarrows}} \Omega^k(M)$$

**Remark:** $(d^*)^2 = 0$

$$d^* := (-1)^{n(k+1)+1} * \circ d \circ *$$

Now, let $M$ be compact (and connected, for simplicity).

- Inner product on each $\Omega^k(M)$:

$$\langle \omega | \eta \rangle := \int_M \omega \wedge *\eta = \int_M \langle \omega | \eta \rangle_g \cdot \omega_g .$$

**Fact:** $d$ and $d^*$ are adjoint w.r. to these inner products:

$$\omega \in \Omega^{k-1}(M),\ \eta \in \Omega^k(M) \Rightarrow \langle d\omega | \eta \rangle = \langle \omega | d^* \omega \rangle .$$

- Finally, define $\Delta := (d + d^*)^2 = dd^* + d^*d$

this is an elliptic operator in each $\Lambda^k T^*M$

the Laplace-Beltrami operator

**Fact:** Let $\omega \in \Omega^k(M)$. TFAE:

(i) $\Delta\omega = 0$ (i.e., $\omega$ is harmonic);

(ii) $d\omega = 0$ and $\underline{d^*\omega}$.

'coclosed'

**Example:** (1) Harmonic functions are precisely constant functions: $\Delta f = 0 \iff df = 0$;

(2) Harmonic top-degree forms: $\Delta(f\omega_g) = 0 \iff$

$$\iff d^*(f\omega_g) = \pm * d * (f\omega_g) = \pm * df = 0 \iff f = \text{const}.$$

**Facts:** (1) [Hodge] $\Omega^k(M) = \rlap{$\underbrace{\phantom{\mathrm{Im}\, d \oplus \mathcal{H}^k(M)}}_{\ker d}$}\mathrm{Im}\, d \oplus \overbrace{\mathcal{H}^k(M) \oplus \mathrm{Im}\, d^*}^{\ker d^*}$,

$\mathcal{H}^k(M)$ is fin.-dim.;

(2) [Corollary] $\mathcal{H}^k(M) \simeq H^k_{dR}(M)$;

(3) $\mathcal{H}^k(M) \xleftrightarrow[\sim]{*} \mathcal{H}^{n-k}(M)$.

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## ② Intrinsically harmonic forms

**Def:** Let $M$ be an oriented compact connected smooth manifold. Then $\omega \in \Omega^k(M)$ is called intrinsically harmonic if $\exists$ a Riem. metric on

M w.r. to which $\omega$ becomes harmonic.

**Observation:** $\omega$ must be closed, and we're looking for a metric making it coclosed.

**Example:** (1) Intrinsically harm. 0-forms are precisely const. f-ns.
(2) Intrinsically harm. n-forms are precisely non-vanishing forms (and zero).

What about forms of intermediate degrees? Do they allow an intrinsic description? **Observ.:** $H^k(M,\mathbb{R})=0 \Rightarrow$ no intr. harm. k-forms

**Guiding example: Riemann surfaces**

Let M be a surface (oriented) of genus g.
Let $\omega \in \Omega^1(M)$ s.th. $\exists$ a Riem. metric on M making $\omega$ harmonic. We have an orientation +

+ a (conf. class of) Riem. metric(s) $\Rightarrow$ we have a complex structure on $M$.

- $\omega$ is real $\Rightarrow \eta = \omega - i I^* \omega = \omega + i * \omega$ is a $(1,0)$-form;
- $\omega$ is harmonic $\Longleftrightarrow \eta$ is holomorphic

But then $(\eta)$ is a canonical class, so $\deg(\eta) = 2g-2$.

**Conclusion:** If $\omega$ is intrinsically harmonic, $\# Z(\omega) \leq 2g-2$.

counted without multiplicities, for it's not clear how to count mult-s for a smooth 1-form.

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In general, let $M$ be an oriented closed smooth $n$-manifold, and let $\omega \in \Omega^k(M)$.

or 'with only nondegenerate zeroes'

- $\omega$ is called Morse if it is transversal to the zero section of $\Lambda^k T^* M \longrightarrow M$;
- $\omega$ is locally intrinsically harmonic if $\exists$ a nbhd $V \supseteq Z(\omega)$ s.th. $\omega|_V$ is intrinsically harmonic;

- $\omega$ is transitive if $\forall\, p \in M \setminus Z(\omega)$ $\exists$ a prop. emb. $k$-dim. subm. $N_p \subseteq M$ s. th. $i^*_{N_p}\omega$ is an orientation form on $N_p$ ($N_p$ must lie in $M \setminus Z(\omega)$ then).

<u>Theorem (Calabi, 1969; Honda, 1996)</u>: If $\omega$ is a closed Morse 1- or $(n-1)$-form on $M$, then it is intrinsically harmonic if and only if it is
(i) locally intrinsically harmonic, and
(ii) transitive.

## ③ Intrinsically harmonic 1-forms

From now on, let $\omega$ stand for a closed 1-form on $M$. Denote $Z = Z(\omega)$, $U := M \setminus Z$.

<u>Observation</u>: $\operatorname{Ker}\omega$ is a distribution of hyperplanes on $U$. It is integrable:

$X, Y \in \Gamma(\operatorname{Ker}\omega) \rightsquigarrow d\omega(X,Y) = X\cdot\langle\omega,Y\rangle - Y\cdot\langle\omega,X\rangle - \langle\omega,[X,Y]\rangle$.

$0$

Let $\mathcal{F}$ denote the corresponding foliation of $U$ by hypersurfaces.

Now, assume $\omega$ is Morse ($\Rightarrow Z$ is a finite set)

## Local intrinsic harmonicity

Let $p \in Z$, $\omega = df$ around $p$. Then $f$ is Morse. Suppose the index of $f$ at $p$ is $k$.

**Lemma:** $\omega$ is loc. intrinsically harmonic at $p \Longleftrightarrow k \neq 0, n$.

**Proof:** ($\Longrightarrow$) Assume the converse. Then we can write $f = \pm[(x^1)^2 + \dots + (x^n)^2]$. Use the maximum principle.

($\Longleftarrow$)
$$f = k \cdot \sum_{i=1}^{n-k} (x^i)^2 - (n-k) \sum_{i=n-k+1}^{n} (x^i)^2 \Longrightarrow$$

$$\omega = 2 \cdot k \sum_{i=1}^{n-k} x^i dx^i - 2(n-k) \sum_{i=n-k+1}^{n} x^i dx^i \Longrightarrow$$

w.r.t. $\bar{g}$ on $\mathbb{R}^n$

$$* \omega = 2 \cdot k \sum_{i=1}^{n-k} (-1)^{i-1} x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n - 2(n-k) \sum_{i=n-k+1}^{n} (-1)^i x^i dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^n \Rightarrow$$

$$d * \omega = 2 \cdot k(n-k) dx^1 \wedge \dots \wedge dx^n - 2 \cdot k(n-k) dx^1 \wedge \dots \wedge dx^n = 0,$$

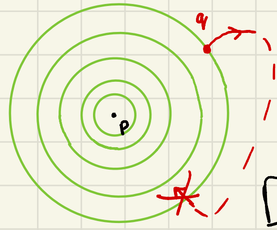
so $\omega$ is loc. intr. harm. at $p$. $\square$

# Transitivity

this is specific to 1-forms and doesn't hold true for (n−1)-forms

**Lemma:** Transitivity ⇒ local intrinsic harm.

**Proof:** Assume $\exists\, p \in Z$ of index 0 or $n$. Then, around $p$, $\mathcal{F}$ looks like this:




$\square$

**Def:** A smooth path $\gamma : I \to U$ is called ==$\omega$-positive== if $\forall\, t \in I$ $\omega(\dot{\gamma}(t)) > 0$.

**Def-n:** Let $p \in U$.

The ==upland== ==$C^+(p)$== $=$

$= \{ q \in U \mid \exists$ a smooth $\omega$-positive path from $p$ to $q \}$.

**TFAE:**
- transitivity;
- $\forall\, p \in U$, $C^+(p) \ni p$;
- $\forall\, p \in U$, $C^+(p)$ is dense in $U$ (or $M$);
- $\forall\, p \in U$, $C^+(p) = U$.

**Facts:** (1) $C^+(p)$ is open and is a union of leaves of $\mathcal{F}$;

(2) $\partial C^+(p)$ (in M) is a union of some leaves of $\mathcal{F}$ + some points of $Z$.


**Proof of the theorem (1-forms, ⇒, idea):** Suffices to prove that for any $p \in U$, $\partial C^+(p) \subseteq Z$! Assume the converse. $\partial C^+(p)$ defines a class in $H_{n-1}(M, \mathbb{Z})$, this class is zero. But $\int_{\partial C^+(p)} * \omega > 0$, contradiction. $\square$

closed form

**Idea for '$\Leftarrow$':** Find a desirable Riem. metric in a nbhd of $Z$, use transitivity to find a desirable metric on $U$, glue the two carefully.

**Fact:** The set of intrins. harmonic 1-forms is $C^1$-open in the set of Morse 1-forms.

**Theorem:** Morse assumption in the previous th-m can be dropped for 1-forms!

**Idea of the proof:** $\Leftarrow$ Gluing, $\Rightarrow$ Poincaré-recurrence th-m.

**Poincaré-recurrence th-m:** Let $(X, \mathcal{A}, \mu)$ be a measure space of finite measure, and let $\mathbb{R} \overset{\theta}{\curvearrowright} X$ be a measure-preserving flow (dynamical system). Let $A \in \mathcal{A}$. Then almost any point of $A$ 'returns to $A$ inf. many times':

$$\mu(A) = \mu(\{x \in A \mid \forall t \in \mathbb{R}_{>0} \ \exists t' > t \ \text{s.th} \ \theta_{t'}(x) \in A\}).$$

In our case, take $X = M$, $\mathcal{A} = \mathrm{Bor}(M)$, $\mu = \mu_g$.

**Fact**: (1) $\exists\, X \in \mathfrak{X}(M)$ s.th. $i_X \omega_g = \star_g \omega$;

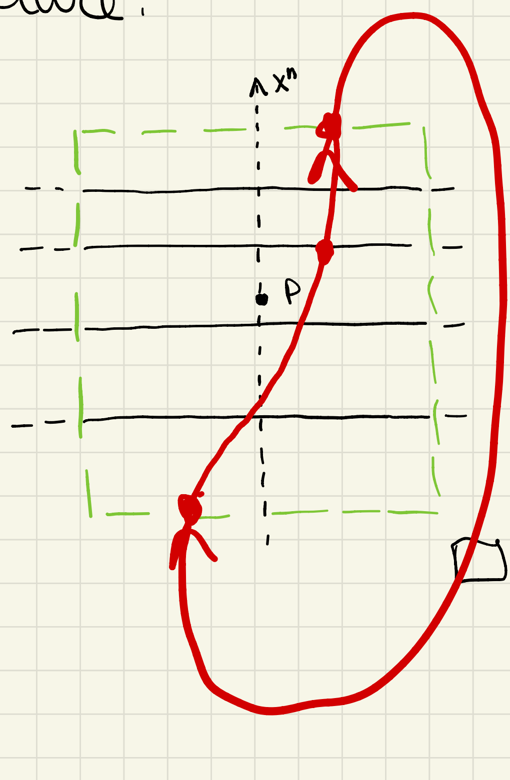
(2) $X$ is transversal to $\mathcal{F}$ on $U$.

Cartan magic f-la:

$$\mathcal{L}_X \omega_g = i_X \underbrace{\cancel{d\omega_g}}_{0} + d i_X \omega_g = d \star \omega = 0,$$

so the flow $\theta$ of $X$ is measure-preserving.

The next is a picture:



## (4) Intrinsically harm. $k$-forms, $1<k<n-1$

**Good news:** Let $\omega \in \Omega^2(M)$ be a symplectic form. Then $\omega$ is intrinsically harmonic.

**Proof:** We can find a compatible Riem. metric $g$ making $(M,\omega)$ into an almost Kähler manifold: $(M, g, \omega, I)$.

**Claim:** $\omega$ is $g$-harmonic.

Indeed, take a good local orthonormal local frame $E_1, E_2, \dots, E_{2n}$ s. th. $\omega = \varepsilon^1 \wedge \varepsilon^2 + \dots + \varepsilon^{2n-1} \wedge \varepsilon^{2n}$.

Then $* \omega = \varepsilon^3 \wedge \dots \wedge \varepsilon^{2n} \pm \varepsilon^1 \wedge \varepsilon^2 \wedge \hat{\varepsilon}^3 \wedge \hat{\varepsilon}^4 \wedge \dots \wedge \varepsilon^{2n} \pm$

$\pm \dots = \frac{\omega^{n-1}}{(n-1)!}$.

But $d\omega = 0 \Rightarrow d(\omega^{n-1}) = 0 \Rightarrow d{*}\omega = 0$. $\square$

**Bad news:** loc. intrins. harm. + trans. $\not\Rightarrow$

$\not\Rightarrow$ intrinsic harm.

**Bad example**: $M \xrightarrow{\pi} S^2$ a nontrivial $S^2$-fiber bundle. Ref.: Steenrod, Topology of Fiber Bundles (1951)

- Take $\omega \in \Omega^2(M)$ to be a pullback of a volume form on $S^2$ (by $\pi$). Clearly, $\omega$ is of const. rank 2 (its kernels are tangent spaces to the fibers of $\pi$).
- **Fact**: $\pi$ admits global sections through any point of $M$. $\Rightarrow$ $\omega$ is transitive.
- Assume that $\omega$ is harmonic for some $g$ on $M$. Denote $*_g \omega =: \eta$. This is a closed 2-form on $M$.
- $\eta$ is of const. rank 2. Look at its kernel distribution. It is integrable by an argument similar to the one before. Take any of its leaves $L$. It is transversal to the fibers of $\pi$ $\Rightarrow$

$\Rightarrow L \xrightarrow{\pi} S^2$ is a submersion, hence a covering map $\Rightarrow$ a diffeomorphism.
So $M$ is foliated by such $L \simeq S^2 \Rightarrow$
$\Rightarrow M$ must be trivial, contrad.

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Ref.: (1) E. Calabi, An intrinsic characterization of harm. 1-forms (1969)
(2) E. Volkov, Characterization of intr. harm. forms (2006)
(3) K. Honda, On harmonic forms for generic metrics (1996)
(4) N. Steenrod, Topology of fiber Bundles (1951).